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ON THE SCREENING OF STATIC ELECTROMAGNETIC FIELDS IN HOT QED PLASMAS

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Abstract

We study the screening of static magnetic and electric fields in massless quantum electrodynamics (QED) and massless scalar electrodynamics (SQED) at temperature T. Various exact relations for the static polarisation tensor are first reviewed and then verified perturbatively to fifth order (in the coupling) in QED and fourth order in SQED, using different resummation techniques. The magnetic and electric screening masses squared, as defined through the pole of the static propagators, are also calculated to fifth order in QED and fourth order in SQED, and their gauge-independence and renormalisation-group invariance is checked. Finally, we provide arguments for the vanishing of the magnetic mass to all orders in perturbation theory.

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1 Introduction

The simplest and probably best known manifestation of many-body effects in electromagnetic plasmas is Debye screening : the usual Coulomb potential between two static charges in a vacuum transforms, in the presence of a plasma, into a Yukawa potential, $V(r) \sim \mathrm{e}^{-m_D r}/r$. The scale m_D is called the electric (Debye) screening mass. For a plasma at high temperature T (much larger than the electron mass), $m_D^2 = e^2T^2/3$ to lowest order in the coupling constant[1].

The above-mentioned relationship between the static potential and the screening mass is usually established within the approximation of linear response (see [2] and section 2), whereby one calculates the potential between two arbitrarily weak *external static* sources q_1 and q_2 separated by $\vec{r}$,

$$V(r) = q_1q_2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2 + \Pi_L(0,p)} = \frac{q_1q_2}{2\pi r} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi i} \frac{p\, e^{ipr}}{p^2 + \Pi_L(0,p)}\,, \tag{1.1}$$

with $\Pi_L(0,p) = -\Pi_{00}(0,p)$, $\Pi_{00}(0,p)$ being the static ($p_0 = 0$) electric polarisation tensor (our conventions and notations are summarized in section 2). The integral over p may be performed by closing the integration path in the upper half of the complex p plane. When the external charges are widely separated ($r \to \infty$), the behaviour of the potential is dominated by the singularity of $D_{00}(0,p) \equiv -1/(p^2 + \Pi_L(0,p))$ which lies closest to the real axis. To leading order, this is a pole occuring for $p^2 = \Pi_{00}^{(2)}(0, p \to 0) = -e^2T^2/3$, $\Pi_{00}^{(2)}$ being the one-loop polarization tensor.

One of the main objectives of this paper is to study, within the definition (1.1), corrections to this leading ($\sim e^2T^2$) Debye mass, and in the process clarify several issues which arise. We shall verify in a perturbative calculation that the dominant singularity of D_{00} remains a pole on the imaginary axis. This allows us to define m_D as the solution of [3]

$$m_D^2 = -\Pi_{00}(0,p)\mid_{p^2=-m_D^2}\,. \tag{1.2}$$

We shall perform the computation in eq.(1.2) up to order e^5 in usual (spinor) electrodynamics (QED), and up to order e^4 in scalar electrodynamics (SQED). In both cases, this corresponds to the first two non-trivial corrections above the "hard thermal loop" approximation[4, 5], and is an extension of previous lower order calculations. As we shall see, the Debye mass obtained in this way is renormalisation group invariant to the required order. General arguments[6] also indicate the gauge-independence of propagator poles and this too will be demonstrated in our computations. At this point it is worth noting that, though we have introduced the screening mass via the potential above, the same

mass m_D controls the exponential decay of other interesting gauge-invariant correlators such as that of static electric fields $\langle E_i(x)E_j(y)\rangle$.

Solving eq. (1.2) requires the knowledge of $\Pi_{00}(0,p)$ for $p \sim m \sim eT$. Thus we are led to study first the simpler object $\Pi_{00}(0,0)$, up to the order of interest. In massless QED, this quantity has been already computed to order e^5, but only indirectly, by using the relation[7, 2]

$$\Pi_{00}(0, p \to 0) \;=\; -\, e^2\,\frac{\partial^2 P}{\partial \mu_e^2}, \tag{1.3}$$

(where P is the plasma pressure and μ_e is the chemical potential for the charged particles), together with an old result for the pressure[8]. Notice that, because of the explicit factor of e^2 on the right-hand-side (r.h.s.) of eq. (1.3), the result for the pressure is only required to third order (but for finite chemical potential). Still, since the derivation of eq. (1.3) is a formal one (see section 2.2), it is of interest to verify the identity explicitly in some cases to show that it is not invalidated, for example, by potential problems such as infrared divergences. This, we shall do in the case of QED: we shall compute the left-hand-side (l.h.s.) of (1.3) to fifth order (and at zero μ_e), and we shall check that it agrees with known results for the r.h.s. In the case of SQED, $\Pi_{00}(0,0)$ has been previously computed up to order e^3[9] (see also Ref. [10]); we will present in this paper the order e^4 correction.

Another objective here is to look into the screening of static magnetic fields. It is known that such fields are not screened in ordinary plasmas. Indeed, one may rely on Ward identities, together with the (exact) Dyson-Schwinger equation to show that

$$\Pi_{ij}(0, p \to 0) = 0\,, \tag{1.4}$$

(see Sect. 2.2 for more details). However, in order to guarantee the absence of perturbative singularities beyond that at $p^2 = 0$ in the correlator of magnetic fields, one needs the stronger result $\Pi_{ij}(0, p \to 0) = \mathcal{O}(p^2)$. This will be verified explicitly up to fifth (respectively, fourth) order in perturbation theory for QED (respectively, SQED). The perturbative arguments can be extended to an all-order proof, to be detailed in Sect. 3.1 in the case of QED.

As in all higher order perturbative calculations at nonzero temperature, a systematic determination of m_D requires a resummation of the large collective plasma effects. For the calculation of dynamical quantities, the resummation involves the procedure developed by Braaten and Pisarski [4] whereby one uses effective vertices and propagators obtained by dressing the bare quantities with "hard thermal loops". The latter are the dominant parts of one-loop amplitudes and express the effects of Landau damping, Debye screening

and collective oscillations [4, 5, 11]. However for the computation of static (zero external energy) Green's functions, like those to be discussed here, the calculations are most natural and convenient in the imaginary time formalism, without analytical continuation to real time. Then the general programme of Refs. [4] reduces to the well known Gell-Mann-Bruekner [12] resummation studied many years ago in the nonrelativistic context and subsequently extended to the relativistic regime [8, 13, 14].

For static calculations in the imaginary time formalism (which we use exclusively in this paper), the resummation concerns only the internal lines with zero Matsubara frequencies (also referred as *static* lines), and consists in dressing these lines with the corresponding screening thermal masses. No vertex resummation will be needed: in QED, there are no vertex in which all lines are soft; in SQED, there are no hard thermal loops beyond the two-point functions [4, 10]. The non-static internal propagators need not be resummed since the corresponding Matsubara frequencies ensure an infrared cut-off of the order of T, relative to which all the thermal corrections are perturbative. This relative simplicity of resummation for static quantities allows different approaches to the detailed calculations. At low orders one can in general perform the resummation of diagrams by "inspection"[13]. In static QED calculations, a "resummation by inspection" (let us call this method (a)) is even feasible at very high orders [15] because the fermion lines are always hard in imaginary time and so do not require dressing. On the other hand, in theories with self-interacting bosonic fields more efficient and practical methods of higher order calculations are required which perform the equivalent resummation, of which there are at least three: (b) truncation of the full skeleton expansion[8, 14]; (c) the use of rearranged lagrangians incoporating the screening masses[16, 4, 17, 18] and, (d) the use of dimensionally reduced effective lagrangians obtained by systematically integrating out of the heavy modes [20, 21] (and references therein). In this paper we will employ resummation by inspection, method (a), for the QED calculations and the truncation of Schwinger-Dyson equations (method (b) above) for SQED. By way of comparison, we also discuss the calculation in SQED using the effective lagrangian, method (d). Method (c)will not be used in this paper but a recent discussion at high orders may be found in Ref.[19].

As a result of the resummation, the perturbative expansion for the electric mass involves *odd* powers of the coupling strength, that is, it is not analytic with respect to e^2. This becomes apparent in spinor QED only at the order e^5, but is already manifest at the order e^3 in scalar QED[9], as well as in QCD[3]. Odd powers of the coupling occur in the perturbative expansion since, after dressing the soft propagators with the corresponding screening masses, the relevant expansion parameter in the infrared is $e^2(T/m) \sim e$, rather

then e^2. From the point of view of its infrared behaviour, SQED is more interesting than spinor QED since it involves interacting bosonic fields. As a consequence, the scalar theory bears more ressemblence to QCD, by exhibiting some of the non-trivial IR structure of this latter theory, but in the (technically less involved) context of an abelian gauge structure. In this respect, SQED serves as a toy model for investigating resummations techniques to be eventually applied in high-temperature QCD (see, e.g., Refs. [22] for such recent applications).

The plan for the rest of the paper is as follows. In Sect. 2 we display our notation and conventions, and derive the relations (1.1), (1.3) and (1.4) in an exact, but formal, manner. Although most of the material in this section may be found scattered in the literature, we have collected it in one place to keep the discussion self-contained. In Sect. 3 we study massless QED in perturbation theory. The highlights in this section are the direct calculation of m_D^2 to order e^5T^2, discussion of its gauge-invariance, and an all-order proof for the vanishing of the static magnetic screening mass. Section 4 is devoted to scalar QED. The notable results obtained here are $\Pi_{\mu\nu}(0, p \to 0)$ and m_D^2 to order e^4T^2, and the vanishing of the magnetic screening mass to the same order. In Sect. 4.6, we re-discuss the results for SQED from the point of view of the effective three-dimensional theory for static fields. This sheds a new light on the resummation, and helps keeping track of the various terms in the diagramatic expansion. We conclude in Sect. 5 with a summary of results and some discussion. Some technical details omitted from Sect. 4 are collected in the Appendices.

2 Notation and General Results

2.1 Conventions

We summarise here our conventions and notation. Unless otherwise stated, all calculations from Sect. 3 onwards will be for the massless theories at zero chemical potential. Ultraviolet divergences are regulated by dimensional continuation ($4 \to D = 4 - 2\epsilon$) and renormalisation is via minimal subtraction. We employ the imaginary time formalism and denote the four-momenta by capitals, $Q_\mu = (q_0, \mathbf{q})$, $q_0 = i\omega_n = in\pi T$, with n even (odd) for bosonic (fermionic) fields. The scalar product is defined with a Minkowski metric, so that $Q^2 = q_0^2 - \mathbf{q}^2$. The measure of loop integrals will be denoted by the following condensed notation:

$$\int [\mathrm{d}Q] \equiv T \sum_{n,even} \int (\mathrm{d}\mathbf{q})\,, \qquad \int \{\mathrm{d}Q\} \equiv T \sum_{n,odd} \int (\mathrm{d}\mathbf{q})\,, \qquad \int [\mathrm{d}Q]' \equiv T \sum_{n\neq 0,even} \int (\mathrm{d}\mathbf{q})\,,$$

where

$$\int (\mathrm{d}\mathbf{q}) \equiv \int \frac{\mathrm{d}^{D-1}q}{(2\pi)^{D-1}}.$$

Note that in Sect.3 we keep the fermions as four-component objects, $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu}$ eventhough the rest of the lagrangian is dimensionally continued.

To simplify the reading of the forthcoming sections, we recall here the tensorial structure of the photon propagator $D^{\mu\nu}$ ($D^{-1}_{\mu\nu} = D^{-1}_{0,\mu\nu} + \Pi_{\mu\nu}$) in the general case, and also in the hard thermal loop approximation. We follow closely the conventions of Ref. [1], and write

$$\Pi^{\mu\nu}(P) = \mathcal{P}_L^{\mu\nu}\Pi_L(p_0,p) + \mathcal{P}_T^{\mu\nu}\Pi_T(p_0,p), \tag{2.1}$$

where $P^\mu = (p_0,\mathbf{p})$, $p = |\mathbf{p}|$, $\hat{p}^i = p^i/p$, and the subscripts L and T refer to longitudinal and transverse directions with respect to the vector $\mathbf{p}$:

$$\begin{aligned} \mathcal{P}_T^{00} = \mathcal{P}_T^{0i} = 0 \qquad\qquad & \mathcal{P}_T^{ij} = \delta^{ij} - \hat{p}^i\hat{p}^j \\ \mathcal{P}_L^{\mu\nu} + \mathcal{P}_T^{\mu\nu} & = \frac{P^\mu P^\nu}{P^2} - g^{\mu\nu}. \end{aligned} \tag{2.2}$$

In the static limit ($p_0 = 0$) the only non trivial components of $D_{\mu\nu}$ are

$$D_{00}(0,\mathbf{p}) = -\frac{1}{\mathbf{p}^2 + \Pi_L(0,p)}, \qquad\qquad D_{ij}(0,\mathbf{p}) = \frac{\delta_{ij} - \hat{p}_i\hat{p}_j}{\mathbf{p}^2 + \Pi_T(0,p)} + \alpha\,\frac{\hat{p}_i\hat{p}_j}{\mathbf{p}^2}, \tag{2.3}$$

in the covariant gauge with gauge fixing parameter α. Note that $\Pi_L(0,p) = -\Pi_{00}(0,p)$ and $\Pi_T(0,p) = (1/2)\Pi_{ii}(0,p)$. *Please note that for simplicity most of our explicit calculations in Sects. 3 and 4 are in Feynman's gauge $\alpha = 1$, but the gauge-independence of our main results (the propagator poles) will be demonstrated.*

The one-loop polarization tensor for ultrarelativistic gauge plasmas has been computed in Refs. [7, 1]. The dominant contribution in the high-temperature limit (i.e. for external momenta which are small compared to the temperature; e.g. p_0 and p of order eT) is the hard thermal loop, which has the same structure for both abelian and non-abelian plasmas[1, 4, 5]:

$$\Pi^{(2)}_{\mu\nu}(P) = m^2\left\{-g_{\mu 0}g_{\nu 0} + p_0\int\frac{\mathrm{d}\Omega}{4\pi}\,\frac{v_\mu v_\nu}{p_0 - \mathbf{v}\cdot\mathbf{p}}\right\}, \tag{2.4}$$

where $m^2 = e^2T^2/3$ (with $e^2 \to g^2N$ for a SU(N) gauge plasma), $v_\mu \equiv (1, \mathbf{v})$, $|\mathbf{v}| = 1$ and the angular integral $\int \mathrm{d}\Omega$ runs over all the directions of $\mathbf{v}$. We have indicated the order of perturbation theory as a superscript on the polarisation tensor, a convention that we shall systematically use througout. The structure (2.4) has a classical origin, as shown by thesimple kinetic derivation[11] which is briefly reviewed in Appendix A. For soft P, $D_0^{-1} \sim P^2 \sim e^2T^2 \sim \Pi^{(2)}_{\mu\nu}$, and the hard thermal loop must be included in the photon propagator. We denote this by ${}^*D_{\mu\nu}$: ${}^*D^{-1}_{\mu\nu} = D^{-1}_{0,\mu\nu} + \Pi^{(2)}_{\mu\nu}$. In the static limit,

$$\Pi^{(2)}_{00}(0,\mathbf{p}) = -m^2 \qquad\qquad \Pi^{(2)}_{ii}(0,\mathbf{p}) = 0, \tag{2.5}$$

so that

$${}^*D_{00}(0,\mathbf{p}) = -\frac{1}{\mathbf{p}^2 + m^2} \qquad\qquad {}^*D_{ij}(0,\mathbf{p}) = \frac{\delta_{ij}}{\mathbf{p}^2}\,. \tag{2.6}$$

2.2 Exact relations

To derive eq. (1.1), consider the plasma in the presence of weak static external sources with charge density $\rho^{ext}(\mathbf{x})$. The free energy in the presence of the sources is $F = -(1/\beta)\ln \mathrm{Tr}\exp\{-\beta(H + H_1)\}$, where H_1 is the Hamiltonian describing the interaction between the gauge fields and the external sources: $H_1 = \int \mathrm{d}^3x\, \rho^{ext}(\mathbf{x})\, A_0(\mathbf{x})$. We assume that the average values of the gauge fields vanish in equilibrium, i.e. in the absence of sources. Then, to second order in ρ^{ext}, the modification in the free energy reads

$$\begin{aligned} F &= F_0 - \frac{1}{2}\int (\mathrm{d}\mathbf{p})\, \rho^{ext}(\mathbf{p})\, D_{00}(0,\mathbf{p})\, \rho^{ext}(-\mathbf{p}) \\ &= F_0 + \frac{1}{2}\int (\mathrm{d}\mathbf{p})\, \frac{\rho^{ext}(\mathbf{p})\, \rho^{ext}(-\mathbf{p})}{\mathbf{p}^2 + \Pi_L(0,\mathbf{p})}\,, \end{aligned} \tag{2.7}$$

where F_0 is the free energy in the absence of sources, and $D_{00}(0,\mathbf{p})$ is the exact electrostatic propagator in equilibrium. By choosing $\rho^{ext}(\mathbf{x}) = q_1\delta(\mathbf{x}-\mathbf{x}_1) + q_2\delta(\mathbf{x}-\mathbf{x}_2)$, one extracts from the above equation the interaction energy of two isolated charges in the medium ($\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$):

$$V(r) = q_1q_2\int (\mathrm{d}\mathbf{p})\, \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2 + \Pi_L(0,p)}\,, \tag{2.8}$$

which is eq. (1.1).

One way to understand eq.(2.7) is to recall that the free energy $F[\rho^{ext}]$ is the generating functional of connected Green's functions. Alternatively, one can express the free

energy in terms of the average gauge field A_0. This is achieved by performing the Legendre tansform $F'[A_0] = F[\rho^{ext}] - \int d^3x\, \rho^{ext} A_0$. In F', the term quadratic in A_0 involves the inverse propagator:

$$
\begin{aligned}
F' &= F_0 + \frac{1}{2}\int (\mathrm{d}\mathbf{p})\, A_0(\mathbf{p})\, D_{00}^{-1}(0,\mathbf{p})\, A_0(-\mathbf{p}) + \ldots \\
&= -\frac{1}{2}\int (\mathrm{d}\mathbf{p})\, A_0(\mathbf{p})\, (\mathbf{p}^2 + \Pi_L(0,\mathbf{p}))\, A_0(-\mathbf{p}) + \ldots .
\end{aligned}
\tag{2.9}
$$

Thus

$$
\Pi_L(0,0) = -\frac{1}{V}\frac{\delta^2 F'}{\delta A_0^2(\mathbf{p}=0) =}\bigg|_{A_0=0}, \tag{2.10}
$$

where V is the volume of the plasma. We note now that the chemical potential μ enters the calculation of the partition function the same w= ay as A_0 does, that is, it amounts to adding to the Hamiltonian a term $-\mu \int d^3x\, \rho_e(\mathbf{x}) = -\mu\, \rho_e(\mathbf{p}=0)$, where $\rho_e(\mathbf{x})$ is the charge density operator (in QED, $\rho_e(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$). Thus, a change of $A_0(\mathbf{p} = 0)$ is equivalent to a change of $-\mu/e$. Since F' and the pressure are related by the thermodynamic relation $F' = -PV$, eq. (2.10) is the same as eq. (1.3).

In order to establish eq. (1.4), we rely on the exact Dyson-Schwinger equations and on the Ward identities. This will also provide us with an alternative proof of eq. (1.3). Notice first that gauge symmetry ensures the transversality of the polarisation tensor,

$$
P^\mu \Pi_{\mu\nu}(p_0, p) = 0\,, \tag{2.11}
$$

so that

$$
p^i \Pi_{i\nu}(0, p) = 0\,, \tag{2.12}
$$

which implies $\Pi_{i0}(0,p) = 0$ and the transversality of $\Pi_{ij}(0,p)$ in the spatial indices.

For spinor QED, the relevant Dyson-Schwinger equation reads

$$
\Pi_{\mu\nu}(p_0, p) = -e^2 \int \{\mathrm{d}K\}\ \mathrm{Tr}\Big(\gamma_\mu S(P+K)\Gamma_\nu(P+K,K)S(K)\Big)\,, \tag{2.13}
$$

where S is the full fermion propagator, Γ the full vertex, and the trace is over spinor indices. In the limit $p_0 = 0, p \to 0$, eq. (2.13) combined with the Ward identity $\Gamma_\nu(K,K) = \partial S^{-1}(K)/\partial K^\nu$ gives

$$
\Pi_{\mu\nu}(p_0 = 0, p \to 0) = -e^2\, \mathrm{Tr}\, \gamma_\mu \int \{\mathrm{d}K\}\, \frac{\partial S(K)}{\partial K^\nu}\,. \tag{2.14}
$$

In the imaginary-time formalism, the variable $k^0 = i\omega_n + \mu_e$ takes discrete values only. Then, the derivative with respect to k^0 in the above equation is meant to be done on the analytic continuation of $S(k_0)$.

In the calculation of $\Pi_{(00)}$ from the r.h.s. of eq. (2.14), one can replace $\partial/\partial k_0 \to \partial/\partial\mu_e$; then, by noticing that

$$\mathrm{Tr}\,\gamma_\mu \int \{\mathrm{d}K\}\, S(K) \,=\, j_\mu \,=\, g_{\mu 0}\,\rho_e \tag{2.15}$$

is the charge density in equilibrium, and that $\rho_e = \partial P/\partial\mu_e$, we recover the identity (1.3). As for the (i, j) components, they vanish after an integration by parts, in agreement with eq. (1.4).

A similar discussion may be carried on for SQED. The corresponding Dyson-Schwinger equation is illustrated in Fig. 1. To save writing, we consider directly the limit $p_0 = 0, p \to 0$ in these diagrams, and denote $\Pi_{\mu\nu} \equiv \Pi_{\mu\nu}(p_0 = 0, p \to 0)$ in the remaining of this section. It is convenient to combine the two diagrams in Figs. 1.a and 1.b by writing

$$\Pi^a_{\mu\nu} + \Pi^b_{\mu\nu} = -2g_{\mu\nu}\, e^2 \int [\mathrm{d}K]\, S(K) - 2e^2 \int [\mathrm{d}K]\, K_\mu\, \Gamma_\nu(K, K)\, S^2(K) \tag{2.16}$$

where $S(K) = 1/(-K^2 + \Sigma(K))$ is the exact scalar propagator, and $\Gamma_\nu(K, Q)$ is the vertex with one photon and two scalar external lines, with K (Q) denoting the momentum carried by theincoming (outgoing) charged particle. We use the same Ward identity as in spinor QED to rewrite eq. (2.16) as

$$\Pi^a_{\mu\nu} + \Pi^b_{\mu\nu} = -2e^2 \int [\mathrm{d}K]\, \frac{\partial}{\partial K^\nu}\Big(K_\mu S(K)\Big)\,. \tag{2.17}$$

After an integration by parts (with the assumption that $k_i S(K) \to 0$ as $|\mathbf{k}| \to \infty$), we obtain $\Pi^a_{ij} + \Pi^b_{ij} = 0$. As for the electric piece, this is rewritten by introducing a small chemical potential for the charged particles, so that $k^0 \equiv i\omega_n + \mu_e$ and

$$\Pi^a_{00} + \Pi^b_{00} = -2e^2\, \frac{\partial}{\partial\mu_e} \int [\mathrm{d}K]\, \Big(k_0 S(K)\Big)\,. \tag{2.18}$$

The contribution of the remaining diagrams, Figs. 1.c and 1.d, is evaluated as

$$\begin{aligned}\Pi^c_{\mu\nu} + \Pi^d_{\mu\nu} \,=\, 2e^2 \int [\mathrm{d}K] \int [\mathrm{d}Q] \Big\{ & g^{\sigma\rho} S(K)\, D_{\mu\sigma}(K-Q)\, S(Q)\, \Gamma_{\rho\nu}(K-Q, 0, K, Q) \\ & - 2S^2(K)\, \Gamma_\nu(K, K)\, S(Q)\, D_{\mu\sigma}(K-Q)\, \Gamma^\sigma(Q, K) \Big\}\end{aligned} \tag{2.19}$$

where $\Gamma_{\mu\nu}$ is the vertex between two photons and two charged scalars. This vertex satisfies

$$\Gamma_{\rho\nu}(K-Q, 0, K, Q) \,=\, -e\, \left(\frac{\partial}{\partial K^\nu} + \frac{\partial}{\partial Q^\nu}\right) \Gamma_\rho(K, Q)\,. \tag{2.20}$$

By using the Ward identities above, one obtains $\Pi^c_{ij} + \Pi^d_{ij} = 0$ and

$$\Pi^c_{00} + \Pi^d_{00} = -2e^3\, \frac{\partial}{\partial\mu_e} \int [\mathrm{d}K] \int [\mathrm{d}Q]\, \Big(S(K)\, S(Q)\, D_{0\rho}(K-Q)\, \Gamma^\rho(K, Q)\Big)\,. \tag{2.21}$$

We recognize in the r.h.s. of eqs. (2.18) and (2.21) the average electric charge density $\rho_e = i(\phi^\dagger \partial_0 \phi - (\partial_0 \phi^\dagger)\phi) - 2e\, A_0 \phi^\dagger \phi$ expressed in terms of exact propagators and vertices. Thus

$$\Pi_{00} \equiv \Pi_{00}^a + \Pi_{00}^b + \Pi_{00}^c + \Pi_{00}^d \;=\; -e^2 \frac{\partial \rho_e}{\partial \mu_e} \;=\; -e^2 \frac{\partial^2 P}{\partial \mu_e^2}\,. \tag{2.22}$$

It is interesting to observe that the vanishing of Π_{ij} is obtained by independent cancellations of the diagrams 1.a and 1.b among themselves, and of the diagrams 1.c and 1.d among themselves.

Note that the manipulations above are formal in the sense that we have left aside the question of the regularization of UV divergences, as well as possible IR problems. We will not address these questions in general here but in the following sections we shall go through an explicit perturbative verification.

3 Perturbative QED

3.1 The static-infrared limit

We now begin our explicit calculations by considering in this subsection the object $\Pi_{\mu\nu}(0, p \to 0)$ up to fifth order (e^5). The explicit computations here have been performed in the Feynman gauge. At one loop, a standard calculation gives

$$\begin{aligned} \Pi^{(2)}_{\mu\nu}(0, p \to 0) &= (e\mu^\epsilon)^2 \int \{dK\}\, \frac{\mathrm{Tr}\,(\gamma_\mu \not{K} \gamma_\nu \not{K})}{K^4} \\ &= -4(e\mu^\epsilon)^2 \int \{dK\} \frac{g_{\mu\nu} K^2 - 2K_\mu K_\nu}{K^4}\,, \end{aligned} \tag{3.1}$$

with $\not{K} \equiv \gamma_\rho K^\rho$. As usual, μ is the mass scale introduced by dimensional regularisation, and the gauge coupling e is dimensionless. Within dimensional regularisation, an integration by parts in the $\mathbf{k}$ integral gives

$$\int \{dK\} \frac{\mathbf{k}^2}{K^4} = \frac{-(D-1)}{2} \int \frac{\{dK\}}{K^2}\,. \tag{3.2}$$

Hence one deduces from (3.1),

$$\Pi^{(2)}_{ij}(0, p \to 0) = 0\,, \tag{3.3}$$

and

$$m^2 \equiv -\,\Pi^{(2)}_{00}(0, p \to 0) \;=\; -4(e\mu^\epsilon)^2 (2-D) \int \frac{\{dK\}}{K^2} \tag{3.4}$$

$$= \frac{e^2 T^2}{3}\,, \tag{3.5}$$

where the limit $D \to 4$ has been taken in the second line. In the rest of this paper, this limit will often be taken implicitly in our final results. The fermionic integral occurring above is a special case of

$$f_n \equiv \int \frac{\{d\,Q\}}{(Q^2)^n} = (2^{2n+1-D} - 1)\; b_n\,, \tag{3.6}$$

where

$$\begin{aligned} b_n &\equiv \int \frac{[d\,Q]}{(Q^2)^n} \\ &= T^{D-2n}\,\frac{2\,(-1)^n\,\pi^{\frac{D-1}{2}}}{(2\pi)^{2n}\,\Gamma(n)}\,\zeta(2n+1-D)\;\Gamma\left(\frac{2n+1-D}{2}\right)\,, \end{aligned} \tag{3.7}$$

and $\zeta(x)$ is Riemann's *zeta*-function. The quickest way to evaluate the b_n is to first integrate over the momenta and then perform the frequency sum. The integral f_n is then obtained using the following trick: consider the sum $b_n + f_n$ and rescale the momenta there by a factor of 2, giving $b_n + f_n = 2^{2n+1-D} b_n$, from which the quoted result follows. Note that, because of the dimensional continuation, the Matsubara mode $q_0 = 0$ does not contribute to the integral in eq. (3.7).

The next contribution is of order e^4 and comes from the two loop diagrams shown in Fig. 2. Each of these diagrams is quite complicated, even in the static IR limit, but their sum is remarkably simple and is given by

$$\Pi^{(4)}_{\mu\nu}(0, p \to 0) = 4(e\mu^\epsilon)^4(D-2)(\Pi^{(4o)}_{\mu\nu} + \Pi^{(4e)}_{\mu\nu}) \tag{3.8}$$

with

$$\Pi^{(4o)}_{\mu\nu} = \int \{dKdR\}\frac{K_\mu R_\nu + K_\nu R_\mu}{K^4 R^4}\,, \tag{3.9}$$

$$\Pi^{(4e)}_{\mu\nu} = (b_1 - f_1)\int \{dK\}\left(\frac{g_{\mu\nu}}{K^4} - \frac{4K_\nu K_\mu}{K^6}\right)\,. \tag{3.10}$$

Note that eq. (3.9) vanishes at zero chemical potential (at nonzero chemical potential it only contributes to Π_{00}). Using integration by parts, as in (3.2), one obtains

$$\Pi^{(4)}_{ij}(0, p \to 0) = 0\,, \tag{3.11}$$

and

$$\Pi^{(4)}_{00}(0, p \to 0) \;=\; 4(e\mu^\epsilon)^4\,(D-2)(D-4)(b_1 - f_1)f_2 \;=\; \frac{e^4 T^2}{8\pi^2}\,. \tag{3.12}$$

Notice the cancellation between the UV divergencies of eqs. (3.9) and (3.10) in their sum (3.12) (the quantity $(D-4)f_2$ being finite for $D \to 4$). This cancellation is necessary since the contribution from the UV counterterm diagrams mutually cancel (see later).

Consider now the behaviour of the magnetostatic polarization tensor for small, but non-vanishing, momenta. We shall argue here that $\Pi_{ij}(0, p \to 0) = \mathcal{O}(p^2)$ to all orders in perturbation theory. To see this, consider an arbitrary graph contributing to $\Pi_{ij}(0, p)$. Label the photon lines in the graph by a set of independent loop momenta so that the external momentum $\mathbf{p}$ only flows along the fermion lines. Since in imaginary time the fermionic propagators are infrared safe, they can be expanded with respect to the soft momentum $\mathbf{p}$. Then the external momentum appears only in the numerator and rotational symmetry ensures that the terms with odd powers of $\mathbf{p}$ vanish. Hence, $\Pi_{ij}(0, p)$ is analytic in p^2 for small p, so that $\Pi_{ij}(0, p \to 0) = \mathcal{O}(1) + \mathcal{O}(p^2)$. By also using eq. (1.4), we conclude that $\Pi_{ij}(0, p)$ vanishes at least as p^2 as $p \to 0$.

The proof in the last paragraph is rigorous only when applied to diagrams without photon self-energy insertions since these latter cause power-like infrared divergences along internal lines due to the nonvanishing of $\Pi_{00}(0, 0)$, thus making the arguments formal. For diagrams with self-energy insertions, the arguments of the last paragraph must be applied not to individual diagrams but to sums of similar diagrams which result in a new effective graph with dressed photon propagators. The conclusion then is as before, namely $\Pi_{ij}(0, p \to 0) = \mathcal{O}(p^2)$. Finally, note that the proof is independent of the fermion mass.

Beyond fourth order we begin to get contributions nonanalytic in e^2. The e^5 term comes from dressing the zero mode of the photon propagators, as shown in Figs. 3 (cf. [15]). The nonzero modes of the photon line (like the modes of the fermion lines) are cut off in the infrared by the scale T and dressing those modes just gives the usual perturbative corrections (which are analytic with respect to e^2). Consider now a static internal photon line: inserting the *electric* polarisation tensor along this line causes infrared divergences which can be summed up, with the result that the bare electrostatic propagator gets replaced by ${}^*D_{00}(0, \mathbf{q}) = -1/(\mathbf{q}^2 + m^2)$, $m^2 = e^2T^2/3$ (recall eq. (2.6)). On the other hand, insertions of the static *magnetic* polarisation tensor give only perturbative corrections, as one can verify from power counting by using $\Pi^{(2)}_{ij}(0, \mathbf{q}) = \mathcal{O}(q^2)$ (also recall that $\Pi_{0i}(0, q) = 0$ to all orders so that there are no corresponding insertions).

The fifth order contribution is then obtained from the diagrams in Fig. 3, in which the internal photon line is static and is associated with the dressed propagator ${}^*D_{\mu\nu}(0, \mathbf{q})$, eq. (2.6). It is then convenient to write

$$\begin{aligned} {}^*D_{\mu\nu}(0, \mathbf{q}) &= D_{0,\mu\nu}(0, \mathbf{q}) + \frac{m^2 g_{\mu 0} g_{\nu 0}}{\mathbf{q}^2(\mathbf{q}^2 + m^2)} \\ &\equiv D_{0,\mu\nu}(0, \mathbf{q}) + {}^*d_{\mu\nu}(0, \mathbf{q}), \end{aligned} \tag{3.13}$$

and to observe that it is the piece ${}^*d$ which is responsible for the e^5 contribution. In

fact, to order e^5, a further approximation can be done. This consists in neglecting the q dependence along the fermion lines. Then, the q integral decouples and is easily evaluated: it is of order $\int \mathrm{d}^3q\, m^2/q^2(q^2+m^2) \sim m \sim e$, and, since one has a factor of e^4 from the four vertices, the net result is of order e^5. The forgotten q-dependence along the fermion lines only gives a subleading contribution because the fermion lines are IR safe, so that one may expand out the q dependence and do the usual power counting.

Thus the exact fifth order contribution comes from the sum of the diagrams in Fig. 3 when ${}^*d(Q)$ is used for the photon lines and the Q dependence along the fermion lines is ignored. We get:

$$\Pi^{(5)}_{\mu\nu}(0,0) = -4(e\mu^\epsilon)^4\, m^{D-3} T \int \frac{(dx)}{x^2(x^2+1)} \int \frac{\{dK\}}{K^8} S_{\mu\nu}\,, \tag{3.14}$$

where

$$\begin{aligned} S_{\mu\nu} \;&=\; (4K_\mu K_\nu - 2g_{\mu\nu}K^2)(K^2+2k^2) + K^4(2g_{\mu 0}g_{\nu 0} - g_{\mu\nu}) \\ &\quad -8k_0K^2(g_{\mu 0}K_\nu + g_{\nu 0}K_\mu) + 16K_\mu K_\nu(K^2+k^2)\,. \end{aligned} \tag{3.15}$$

Again using manipulations as in (3.2) gives

$$\Pi^{(5)}_{ij}(0, p\to 0) = 0\,, \tag{3.16}$$

and

$$\begin{aligned} \Pi^{(5)}_{00}(0,p\to 0) \;&=\; 4(e\mu^\epsilon)^4\, m^{D-3} T\,(D-2)(D-4) f_2 \int \frac{(dx)}{x^2(x^2+1)} \\ &=\; \frac{-e^5T^2}{4\pi^3\sqrt{3}}\,. \end{aligned} \tag{3.17}$$

In summary we have shown for massless QED in the Feynman gauge at temperature T and zero chemical potential μ_e,

$$\Pi_{ij}(0,p\to 0) = \mathcal{O}(p^2) + \mathcal{O}(e^6T^2)\,. \tag{3.18}$$

and

$$\Pi_{00}(0,p\to 0) = -T^2\left(\frac{e^2}{3} - \frac{e^4}{8\pi^2} + \frac{e^5}{4\pi^3\sqrt{3}}\right) + \mathcal{O}(e^6T^2)\,. \tag{3.19}$$

A scan of the computations shows that Eq.(3.18) has also been verified for nonzero chemical potential. The result (3.19) — which is the left-hand-side of the identity eq. (1.3) — is in accordance with the r.h.s. of as given in [2]. Also, since the right-hand-side of (1.3)

is gauge-independent, this implies the gauge-independence of (3.19) though we have done our calculations in the simpler Feynman gauge.

The result eq. (3.19) has turned out to be UV finite eventhough we seem to have ignored the UV renormalisation. However recall first that zero-temperature counterterms suffice to render the theory UV finite, and that gauge invariance ensures that the vacuum (i.e. zero-temperature) polarization tensor vanishes in the zero-momentum limit. Then, in our calculation, no counterterms were needed for the subdiagrams of the two loop polarization tensor for the following reasons: i) The order e^4 vertex and fermion wave-function counterterm diagrams cancel against each other because of the Ward identity $Z_1 = Z_2$; ii) since we are working in the massless limit, there is no mass counterterm; iii) there is no order e^5 counterterm diagram. On the other hand, UV divergences will make their appearance when the polarization tensor will be considered for non-vanishing momenta in the next section. Similar arguments apply to the SQED calculation in Sect.4, and so we shall not repeat them there.

3.2 Screening masses

Having computed $\Pi_{\mu\nu}(0, p \to 0)$, we are ready to determine the screening masses. To one-loop order, the electrostatic propagator ${}^*D_{00}(0, \mathbf{p}) = -1/(\mathbf{p}^2 + m^2)$ has a simple pole in the upper half of the complex p-plane, occuring at $p = im$. Accordingly, $m_D^2 = m^2 + \mathcal{O}(e^4 T^2)$ (recall eq.(1.2)). The higher-order corrections that we have calculated modify the position of this pole, without changing the analytic structure of $D_{00}(0, \mathbf{p})$. Accordingly, we may define the electric screening mass as the solution of the equation

$$p^2 - \Pi_{00}(0, p) = 0 \tag{3.20}$$

for $p \sim eT$. Since we will only do the calculations up to order e^5, we need the expansion of the one and two loop static polarisation tensors up to the following orders,

$$\Pi_{00}^{(2)}(0, p \sim eT) = -e^2 T^2 \left(a_{20} + \frac{a_{21}\ p}{T} + \frac{a_{22}\ p^2}{T^2} + \frac{a_{32}\ p^3}{T^3} \right) + \mathcal{O}(e^2 p^4 / T^2) \tag{3.21}$$

$$\Pi_{00}^{(4)}(0, p \sim eT) = e^4 T^2 \left(a_{40} + \frac{a_{41}\ p}{T} \right) + \mathcal{O}(e^4 p^2)\,, \tag{3.22}$$

together with $\Pi_{00}^{(5)}(0, 0)$.

From the last subsection we have $a_{20} = 1/3$ and $a_{40} = 1/8\pi^2$. Furthermore $a_{21} = a_{23} = a_{41} = 0$ for the reasons explained after eq. (3.12). We now calculate a_{22} starting

from

$$\Pi_{00}^{(2)}(0,p) = 2(e\mu^{\epsilon})^2 \int \{dK\} \delta_{p_0,0} \left(\frac{2}{K^2} + \frac{4k^2 - p^2}{K^2(K+P)^2} \right) . \tag{3.23}$$

Since k_0 is discrete and of order T, one may expand

$$\frac{\delta_{p_0,0}}{(K+P)^2} = \frac{\delta_{p_0,0}}{K^2} \left(1 + \frac{2\mathbf{k}\cdot\mathbf{p} + \mathbf{p}^2}{K^2} + \frac{(2\mathbf{k}\cdot\mathbf{p} + \mathbf{p}^2)^2}{K^4} + ... \right) . \tag{3.24}$$

Then using (3.24) in (3.23) and simplifying , we get

$$\Pi_{00}^{(2)}(0,p) = \Pi_{00}^{(2)}(0,0) \; - \; \frac{2(e\mu^{\epsilon})^2 p^2}{3}(D-2) f_2 + \mathcal{O}(e^2 p^4/T^2) \, . \tag{3.25}$$

The order e^2p^2 term in (3.25) is UV divergent as $D \to 4$ and this divergence is cancelled by the photon wave-function renormalisation counterterm $\delta Z_3 p^2$, with $\delta Z_3 = -e^2/12\pi^2\epsilon$. Then the renormalized value of a_{22} reads

$$a_{22}^R = \frac{4\mu^{2\epsilon}(1-\epsilon)}{3} f_2 - \frac{1}{12\pi^2\epsilon} \; = \; \frac{1}{12\pi^2} \left(\gamma - 1 + \ln \frac{4\mu^2}{\pi T^2} \right) . \tag{3.26}$$

We summarize the above results by writing ($p \sim eT$)

$$\Pi_{00}(0,p) = -m^2 - a_{22}^R e^2 p^2 + \Pi_{00}^{(4)}(0,0) + \Pi_{00}^{(5)}(0,0) + \mathcal{O}(e^6 T^2) \, . \tag{3.27}$$

The solution $p^2 = -m_D^2$ of eq. (3.20) to fifth order is therefore

$$m_D^2 = T^2 \left(\frac{e^2}{3} - \frac{e^4}{8\pi^2} - \frac{e^4}{36\pi^2} \left[\gamma - 1 + \ln \frac{4\mu^2}{\pi T^2} \right] + \frac{e^5}{4\pi^3\sqrt{3}} \right) + \mathcal{O}(e^6 T^2) \, . \tag{3.28}$$

Here, $e \equiv e(\mu)$ is the running coupling constant defined by the minimal subtraction scheme. Since this satisfies $\mathrm{d}e/\mathrm{d}\ln\mu = e^3/12\pi^2$, it is clear that the r.h.s. of eq. (3.28) is independent of the renormalization scale μ, to order e^5.

Eq. (3.28) is our result for the electric screening mass of massless QED at temperature T and zero chemical potential. Up to order e^4 this coincides with the result of Ref. [3]. As for the magnetic screening mass, this vanishes as expected, since $\Pi_{ij}(0, p \to 0) = \mathcal{O}(p^2)$.

Let us now establish the gauge-independence of our result (3.28). The constants a_{2n} come from the one-loop diagram and so are manifestly gauge-independent. The constant a_{40} is gauge-independent because of the relation (1.3) while the vanishing of a_{41} is a gauge-independent statement since the arguments following eq. (3.12) make no reference to any gauge-choice. Hence the electric screening mass (and similarly the vanishing magnetic screening mass) to fifth order as given by eq. (3.28) is gauge-independent.

4 Perturbative Scalar QED

In this section we consider the electromagnetic interactions of a charged scalar field ϕ described by the Lagrangian ($D_\mu = \partial_\mu + ieA_\mu$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial \cdot A)^2 + (D_\mu \phi)^\dagger (D^\mu \phi) - \frac{\lambda}{4}(\phi^\dagger \phi)^2. \tag{4.1}$$

Since we are interested only in the effects of the electromagnetic interactions, we shall ignore the self coupling of the complex field ϕ, i.e we assume $\lambda \to 0$ in what follows.

In this section, we shall compute the correction of order e^4T^2 to the Debye mass, and we shall verify that $\Pi_{ij}(0, \mathbf{p}) = \mathcal{O}(p^2)$ as $p \to 0$, to the same order. In order to avoid double counting in higher order calculations, we shall refer to the skeleton diagrams displayed in Fig. 1. The corrections to $\Pi_{\mu\nu}$ at various orders will be obtained by expanding to the appropriate order the exact propagators or vertices in these diagrams. As explained before, in this procedure one must keep the thermal masses on the static propagators, while non-static propagators can be perturbatively expanded.

4.1 Leading order results: hard thermal loops

In leading order, the self-energies for the photon and the scalar particle are obtained from the 1-loop diagrams in Figs. 4 and 5 respectively. For soft external momenta (that is, for p_0 and p of order eT), the dominant contributions to these diagrams come from loop momenta of the order of T: these are the hard thermal loops[4, 5]. The photon hard thermal loop was already presented in Sect. 2 (eqs. (2.4)–(2.6)). The hard thermal loop for the scalar self-energy reduces to a (gauge-independent) local mass term:

$$\Sigma^{(2)}(P) = (D-1)\,(e\mu^\epsilon)^2 \int [\mathrm{d}Q]\, S_0(Q) = \frac{e^2T^2}{4} \equiv M^2\,, \tag{4.2}$$

where $S_0(Q) = -1/Q^2$. We define the propagator ${}^*S$ by ${}^*S^{-1} = S_0^{-1} + \Sigma^{(2)}$. In the static limit,

$${}^*S(0, \mathbf{p}) = \frac{1}{\mathbf{p}^2 + M^2}\,. \tag{4.3}$$

Generally, the one-loop results cannot be trusted beyond the hard thermal loop approximation. This is so because, beyond leading order, soft loop momenta start to contribute and the corresponding propagators must include the hard thermal loops. Consider

for example the small momentum behavior of the static one-loop polarisation tensor. It is shown in Appendix B that

$$\Pi_{ii}^{1L}(0,p) = \frac{1}{8}e^2 pT + \mathcal{O}(e^2p^2) \tag{4.4}$$

(see eq. (B.2)). A similar behaviour is obtained for the scalar one-loop self-energy:

$$\Sigma_{1L}(0,p) = M^2 - \frac{1}{4}e^2 pT + \mathcal{O}(e^2p^2). \tag{4.5}$$

These non-analytic (in p^2) contributions arise from the static internal modes. We shall verify in the next subsection that these terms disappear once thermal masses are included in static internal lines. In QED, such a problem does not occur because the corresponding one-loop diagram for $\Pi_{\mu\nu}$ has only fermionic internal lines. Note that for the component Π_{00} no resummation is needed to get the leading low momentum behavior to one-loop order (see eq. (B.6)): because of the vector structure of the electromagnetic interaction, the mode $n = 0$ does not contribute to the Matsubara sum in the contribution of the diagram 4.b to $\Pi_{00}(0,p)$; as for the diagram 4.a, it is momentum independent.

4.2 Next-to-leading order: ring summation

In this subsection, we consider the consequences of the resummation on the one-loop diagrams of Figs. 4 and 5. We isolate the *static* Matsubara mode, since this is the only one which is concerned by the resummation, and we replace the bare propagators by the propagators ${}^*D_{\mu\nu}$ and ${}^*S$ obtained in the hard thermal loop approximation. We thus obtain the dressed one-loop diagrams of Figs. 6 and 7, whose contributions will be denoted by ${}^*\Pi_{\mu\nu}(0,\mathbf{p})$ and ${}^*\Sigma(0,\mathbf{p})$, respectively.

The tadpole diagram in Fig. 6.a gives

$${}^*\Pi^a_{\mu\nu} = -2g_{\mu\nu}\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\left(\frac{1}{\mathbf{k}^2+M^2} - \frac{1}{\mathbf{k}^2}\right)\,. \tag{4.6}$$

The second term inside the brakets in eq.(4.6) substracts that contribution of the static mode which has already been included in the hard thermal loop, eq. (2.4). Note that this term vanishes in dimensional regularisation so we could have as well omitted it. However it is better to keep it here. This will allow us to show that the final results, when appropriate contributions of a given order are added, are both ultraviolet and infrared finite, even in the absence of regularisation. Letting $D \to 4$ in eq. (4.6), we get

$${}^*\Pi^a_{\mu\nu} = \frac{g_{\mu\nu}}{2\pi}\,e^2MT = \frac{g_{\mu\nu}}{4\pi}\,e^3T^2\,. \tag{4.7}$$

The diagram in Fig. 6.b gives a non trivial contribution, of order e^3, only to the magnetic piece $\Pi_{ii}(0,p)$. As $p \to 0$,

$$
\begin{aligned}
{}^*\Pi^b_{ii}(0,0) &= -4\,(e\mu^\epsilon)^2\,T \int (\mathrm{d}\mathbf{k})\,\mathbf{k}^2 \left(\frac{1}{(\mathbf{k}^2+M^2)^2} - \frac{1}{\mathbf{k}^4} \right) \\
&= \frac{3}{2\pi}\,e^2MT = \frac{3}{4\pi}\,e^3T^2,
\end{aligned}
\tag{4.8}
$$

By adding together the contributions (4.7) and (4.8), we obtain the total contribution of order e^3 to the zero-momentum limit of the polarisation tensor[9]:

$$
\Pi^{(3)}_{00}(0,0) = \frac{e^2MT}{2\pi} = \frac{e^3T^2}{4\pi}, \qquad\qquad \Pi^{(3)}_{ii}(0,p\to 0) = 0.
\tag{4.9}
$$

The above correction to $\Pi_{00}(0,0)$ can be understood as a classical correction, as discussed at the end of Appendix A.

Consider now the momentum dependence of ${}^*\Pi_{\mu\nu}(0,p)$. The electric piece is independent of p, since it is entirely given by the tadpole of Fig. 6.a (recall eq. (4.7)). For the magnetic piece, both diagrams in Fig. 6 contribute, and give

$$
{}^*\Pi_{ii}(0,\mathbf{p}) = (e\mu^\epsilon)^2\,T \int (\mathrm{d}\mathbf{k}) \left\{ \frac{2(D-1)}{\mathbf{k}^2+M^2} - \frac{(2\mathbf{k}+\mathbf{p})^2}{(\mathbf{k}^2+M^2)\,((\mathbf{k}+\mathbf{p})^2+M^2)} \right\}.
\tag{4.10}
$$

The presence of the mass $M \sim eT$ in the denominators allows an expansion of the last one with respect to $\mathbf{p}$. After integration over $\mathbf{k}$, only the terms even in $\mathbf{p}$ survive. The small momentum expansion of (4.10) is therefore

$$
{}^*\Pi_{ii}(0,\mathbf{p}) = \frac{e^2p^2}{12\pi}\,\frac{T}{M} \left\{ 1 + \mathrm{c}_1\,(p/M)^2 + \mathrm{c}_2\,(p/M)^4 + \ldots \right\},
\tag{4.11}
$$

where the c_i's are constant coefficients. Note that there is no term linear in p, contrary to the pure one-loop result of eq. (B.1). As $p \to 0$, the leading term is proportional to $e^2(T/M)p^2 \sim ep^2$. For $p \sim eT$, all the terms in the r.h.s. are of order e^3T^2, and the integral in (4.10) should be computed exactly, with the following result:

$$
{}^*\Pi_{ii}(0,\mathbf{p}) = \frac{e^2MT}{2\pi} \left\{ \frac{4M^2+p^2}{2pM} \arctan\frac{p}{2M} - 1 \right\}.
\tag{4.12}
$$

For small momenta $p \lesssim eT$, this equation gives the leading infrared behaviour of the static magnetic polarization operator (the non-static modes in the one-loop diagrams in Fig. 4 contribute only to order e^2p^2). For large momenta, $p/M \gg 1$, ${}^*\Pi_{ii}(0,\mathbf{p}) \to e^2pT/8$, as for the undressed one-loop result of eq. (4.4).

A similar discussion applies to ${}^*\Sigma(0,\mathbf{p})$, the correction to the static scalar self-energy given by the dressed one-loop diagrams of Fig. 7 (which, we recall, involve only the internal mode with zero frequency). A straightforward computation gives

$$ {}^*\Sigma(0,\mathbf{p}) = (e\mu^\epsilon)^2\, T\int(\mathrm{d}\mathbf{q})\left\{\frac{1}{\mathbf{q}^2+m^2}+\frac{1}{\mathbf{q}^2+M^2}+\frac{D-3}{\mathbf{q}^2}-\frac{2}{\mathbf{q}^2}\frac{\mathbf{p}^2-M^2}{(\mathbf{q}+\mathbf{p})^2+M^2}\right\}, \tag{4.13} $$

in the Feynman gauge. The q-integral can be performed readily, and for $(D-1)\to 3$, one gets

$$ {}^*\Sigma(0,\mathbf{p}) = \frac{e^2MT}{4\pi}\left\{\frac{2(M^2-\mathbf{p}^2)}{Mp}\arctan\frac{p}{M}-\frac{m+M}{M}\right\}, \tag{4.14} $$

which admits the following small-momentum expansion:

$$ {}^*\Sigma(0,\mathbf{p}) = \frac{e^2MT}{4\pi}\left\{\frac{M-m}{M}-\frac{8}{3}(p/M)^2+\mathcal{O}(p^4/M^4)\right\}. \tag{4.15} $$

For high momenta, $p/M \gg 1$, we recover the linear behaviour in p as in eq. (4.5). For $p \lesssim eT$, eq. (4.14) gives the dominant non-trivial momentum behaviour of the scalar self-energy.

4.3 Order e^4: diagrams 1.a and 1.b

Contributions of order e^4T^2 arise from two-loop diagrams in which the static propagators are dressed by thermal masses. Because of the resummation involved in the thermal masses, parts of these two-loop diagrams have already been included in the order e^3 calculation. In order to avoid double counting, we refer to the skeleton diagrams of Fig. 1. We shall give details only for the tadpole diagram, Fig. 1.a:

$$ \Pi^a_{\mu\nu} = -2g_{\mu\nu}\,(e\mu^\epsilon)^2\int[\mathrm{d}K]\,S(K) \equiv g_{\mu\nu}\Pi^a \tag{4.16} $$

where $S(K) = 1/(-K^2+\Sigma(K))$ is the exact scalar propagator, with $\Sigma(K)$ the exact self-energy. In these expressions $K=(2i\pi nT,\mathbf{k})$. When $n\neq 0$, Σ represents a small perturbative correction and only the second term in the expansion

$$ S(K) = -\frac{1}{K^2}\left(1+\frac{\Sigma(K)}{K^2}+\cdots\right) \tag{4.17} $$

is in fact needed to evaluate (4.16) up to order e^4. When $n=0$, infrared divergences render the expansion above meaningless. It is then convenient to expand about the

massive propagator ${}^*S(0,\mathbf{k})$ (eq. (4.3)):

$$
\begin{aligned}
S(0,\mathbf{k}) &= \frac{1}{\mathbf{k}^2+M^2+(\Sigma(0,\mathbf{k})-M^2)} \\
&= \frac{1}{\mathbf{k}^2}+\left(\frac{1}{\mathbf{k}^2+M^2}-\frac{1}{\mathbf{k}^2}\right)-\frac{\Sigma(0,\mathbf{k})-M^2}{(\mathbf{k}^2+M^2)^2}+\cdots
\end{aligned}
\tag{4.18}
$$

When the expansions (4.17) and (4.18) are used in eq.(4.16), the following result is obtained

$$
\begin{aligned}
\Pi^a &= -2\,(e\mu^\epsilon)^2\int[\mathrm{d}K]\,S_0(K) \\
&\quad -2\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\,[{}^*S(0,\mathbf{k})-S_0(0,\mathbf{k})] \\
&\quad +2\,(e\mu^\epsilon)^2\int[\mathrm{d}K]'\,S_0^2(K)\,\Sigma(K) \\
&\quad +2\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\,{}^*S^2(0,\mathbf{k})\left[\Sigma(0,\mathbf{k})-M^2\right] \\
&\quad \cdots
\end{aligned}
\tag{4.19}
$$

where the neglected terms are, at least, of order e^5T^2. The self-energy entering the r.h.s. is the one-loop self-energy, that is, it is obtained from the diagrams in Fig. 5 or, if necessary, from the dressed diagrams of Fig. 7 (see below).

Consider now the different terms in the right hand side of eq. (4.19). We have already evaluated the first two integrals giving respectively the contributions of order e^2 and e^3. For the third integral, it is enough (since $k^0\sim T$) to use the *one-loop* expression of the scalar self-energy Σ, i.e.

$$
2\,(e\mu^\epsilon)^2\int[\mathrm{d}K]'\,S_0^2(K)\,\Sigma_{1L}(K),
\tag{4.20}
$$

where (see Fig. 5)

$$
\Sigma_{1L}(K)=M^2+2\,(e\mu^\epsilon)^2\;K^2\int[\mathrm{d}Q]\,S_0(Q)\,S_0(K+Q),
\tag{4.21}
$$

in the Feynman gauge.

The evaluation of the last term in eq. (4.19) is more involved. It is again necessary to separate the static ($\omega_m=0$) and non-static ($\omega_m\neq 0$) modes in the one-loop diagrams giving $\Sigma(0,\mathbf{k})$ (see Figs. 5 and 7); here, ω_m denotes the Matsubara frequency inside the loop. For the non static modes, bare propagators can be used, as in Fig. 5, and we recover the $m\neq 0$ piece of the one-loop self-energy from eq. (4.21). The corresponding contribution to Π^a reads

$$
\begin{aligned}
&2\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\,{}^*S^2(0,\mathbf{k})\left[\Sigma_{1L}(0,\mathbf{k})-M^2\right]_{m\neq 0} \\
&= -4\,(e\mu^\epsilon)^4\,T\int(\mathrm{d}\mathbf{k})\,\frac{\mathbf{k}^2}{(\mathbf{k}^2+M^2)^2}\int[\mathrm{d}Q]'\frac{1}{\omega_m^2+\mathbf{q}^2}\,\frac{1}{\omega_m^2+(\mathbf{q}+\mathbf{k})^2}.
\end{aligned}
\tag{4.22}
$$

It is not difficult to see that the above contribution is of order $e^4MT \sim e^5T^2$ (in $D = 4$) and therefore can be ignored in our present computation of the order e^4. In fact, eq. (4.22) is precisely of the type already encountered in Sect. 3.1, in computing the contribution of order e^5 to $\Pi_{\mu\nu}(0,0)$ for spinor QED. In particular, its ultraviolet singularity in the limit $D \to 4$ is harmless, since it will be compensated by similar contributions arising from other *mixed* two-loop diagrams (as happens, e.g., in eq. (3.17)). (We characterize as "mixed" any two-loop graph where one of the internal frequencies is non-vanishing, while the other one is zero.) Mixed graphs do not contribute to $\Pi_{\mu\nu}(0,p)$ to order e^4, and will be neglected in what follows. (This applies, in particular, to the mixed graph included in eq. (4.20).)

Consider now the remaining contribution to eq. (4.19), that is,

$$2\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\,{}^*S^2(0,\mathbf{k})\left[{}^*\Sigma(0,\mathbf{k})-M^2\right]_{m=0}. \tag{4.23}$$

This corresponds to two-loop diagrams where *both* the internal frequencies are zero, so that the corresponding propagators are dressed by the hard thermal loops. The integral (4.23) is explicitly written in eq. (4.32) below.

We turn now to the diagram 1.b, which gives

$$\Pi^b_{\mu\nu}(0) = -2\,(e\mu^\epsilon)\int[\mathrm{d}K]\,k_\mu\,\Gamma_\nu(K,K)\,S^2(K)\,. \tag{4.24}$$

To one-loop order, and for $Q = K$, one readily gets $\Gamma_\nu(K,K) = 2e\mu^\epsilon K_\nu + \delta\Gamma^{1L}_\nu(K,K)$, with

$$\delta\Gamma^{1L}_\mu(K,K) = -4\,(e\mu^\epsilon)^3\int[\mathrm{d}Q]\,S_0(Q)S_0(K+Q)\left\{K_\mu + (K_\mu+Q_\mu)K^2S_0(K+Q)\right\}, \tag{4.25}$$

in Feynman gauge. For $k_0 \equiv i\omega_n \neq 0$, the one-loop expressions for the vertex and the scalar self-energy, as given by (4.21) and (4.25), are sufficient to get the order e^4 contribution to eq. (4.24):

$$\begin{aligned}\left[\Pi^{(4b)}_{\mu\nu}(0)\right]_{n\neq 0} &= -2\,(e\mu^\epsilon)\int[\mathrm{d}K]'\,k_\mu\,\delta\Gamma^{1L}_\nu(K,K)\,S_0^2(K)\\ &\quad +8\,(e\mu^\epsilon)^2\int[\mathrm{d}K]'\,k_\mu k_\nu\,\Sigma_{1L}(K)\,S_0^3(K).\end{aligned} \tag{4.26}$$

In the calculation of Π_{00} we only have to consider non vanishing Matsubara frequencies $k_0 \neq 0$, so that $\Pi^{(4b)}_{00}$ is completely determined by eq. (4.26) above. The static mode only contributes to the magnetic sector, and there the resummation of the thermal masses is again necessary. In this case, we expand as follows (compare eq. (4.19)):

$$\begin{aligned}\left[\Pi^{(4b)}_{ii}(0)\right]_{n=0} &= -2\,(e\mu^\epsilon)\,T\int(\mathrm{d}\mathbf{k})\,k_i\,\delta\Gamma_i(\mathbf{k},\mathbf{k})\,{}^*S^2(0,\mathbf{k})\\ &\quad +8\,(e\mu^\epsilon)^2\,T\int(\mathrm{d}\mathbf{k})\,\mathbf{k}^2\,{}^*S^3(0,\mathbf{k})\left[\Sigma(0,\mathbf{k})-M^2\right].\end{aligned} \tag{4.27}$$

Let ω_m denote the internal Matsubara frequency in the one-loop diagrams contributing to $\delta\Gamma_i(\mathbf{k},\mathbf{k})$ or to $\Sigma(0,\mathbf{k})$. The non-static modes ($\omega_m \neq 0$) give rise to mixed two-loop graphs which contribute to eq. (4.27) only to order e^5; they will be ignored in the present calculation. For the static mode ($\omega_m = 0$), the resummation of the thermal masses is compulsory, so that the corresponding self-energy ${}^*\Sigma(0,\mathbf{k})$ and vertex-correction ${}^*\delta\Gamma_i(\mathbf{k},\mathbf{k})$ are determined from dressed one-loop diagrams. The corresponding result for the self-energy has been given in eq. (4.13). For the vertex function, one obtains similarly

$$\begin{aligned} \left[k_i{}^*\delta\Gamma_i(\mathbf{k},\mathbf{k})\right]_{m=0} &= -4\,(e\mu^\epsilon)^3\, T \int (\mathrm{d}\mathbf{q})\, S_0(0,\mathbf{q})\,{}^*S(0,\mathbf{k}+\mathbf{q}) \\ &\qquad \left\{\mathbf{k}^2 - \mathbf{k}\cdot(\mathbf{k}+\mathbf{q})(\mathbf{k}^2 - M^2)\,{}^*S(0,\mathbf{k}+\mathbf{q})\right\} \\ &= (e\mu^\epsilon)\; k_i \left.\frac{\partial^*\Sigma(0,\mathbf{k})}{\partial k_i}\right|_{m=0}. \end{aligned} \tag{4.28}$$

As shown by the last line above, this result is consistent with the Ward identity and with the expression (4.13) of ${}^*\Sigma(0,\mathbf{k})$. After combining eqs. (4.27), (4.13) and (4.28), one obtains the result displayed in eq. (4.39) below.

We now summarize the results obtained in this section. We have

$$\Pi^{(4a)} = A_1 + A_2 + A_3, \tag{4.29}$$

where

$$A_1 = 2\,(e\mu^\epsilon)^2\, M^2 \int [\mathrm{d}K]'\, S_0^2(K)\,, \tag{4.30}$$

$$A_2 = -4\,(e\mu^\epsilon)^4 \int [\mathrm{d}K]' \int [\mathrm{d}Q]'\, S_0(K) S_0(K+Q) S_0(Q)\,, \tag{4.31}$$

$$\begin{aligned} A_3 = 2\,(e\mu^\epsilon)^4\, T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{(\mathbf{k}^2+M^2)^2} \\ \left\{\frac{1}{\mathbf{q}^2+m^2} + \frac{1}{\mathbf{q}^2+M^2} - \frac{2}{\mathbf{q}^2} - \frac{2}{\mathbf{q}^2}\frac{\mathbf{k}^2-M^2}{(\mathbf{q}+\mathbf{k})^2+M^2}\right\}. \end{aligned} \tag{4.32}$$

The first two terms, A_1 and A_2, are obtained from eqs. (4.20) and (4.21). In writing them, we have omitted mixed terms with $\omega_n \neq 0$ and $\omega_m = 0$, since they contribute only to higher orders. The term A_3 follows immediately from eqs. (4.23) and (4.13).

Similarly

$$\Pi_{00}^{(4b)}(0) = B_{00}^1 + B_{00}^2\,, \tag{4.33}$$

with

$$B_{00}^{1} = 8\,(e\mu^{\epsilon})^{2}\,M^{2}\int[\mathrm{d}K]\,k_0^2\,S_0^3(k)\,, \tag{4.34}$$

$$B_{00}^{2} = -12\,(e\mu^{\epsilon})^{4}\int[\mathrm{d}K]\int[\mathrm{d}Q]\,k_0^2\,S_0^2(K)\,S_0(K+Q)\,S_0(Q). \tag{4.35}$$

Finally, for the magnetic contribution we set

$$\Pi_{ii}^{(4b)}(0) = B_{ii}^{1} + B_{ii}^{2} + B_{ii}^{3}\,, \tag{4.36}$$

with

$$B_{ii}^{1} = -2\,(e\mu^{\epsilon})^{2}\,M^{2}\int[\mathrm{d}K]'\,k_i\frac{\partial}{\partial k_i}S_0^2(K)\,, \tag{4.37}$$

$$B_{ii}^{2} = 4\,(e\mu^{\epsilon})^{4}\int[\mathrm{d}K]'\int[\mathrm{d}Q]'\,S_0(Q)\,k_i\frac{\partial}{\partial k_i}\left[S_0(K)S_0(K+Q)\right]\,, \tag{4.38}$$

$$B_{ii}^{3} = -2\,(e\mu^{\epsilon})^{4}\,T^{2}\int(\mathrm{d}\mathbf{k})\int(\mathrm{d}\mathbf{q})\,k^i\frac{\partial}{\partial k^i}\left\{\frac{1}{(\mathbf{k}^2+M^2)^2}\right. \tag{4.39}$$
$$\left.\left[\frac{1}{\mathbf{q}^2+m^2}+\frac{1}{\mathbf{q}^2+M^2}-\frac{2}{\mathbf{q}^2}-\frac{2}{\mathbf{q}^2}\frac{\mathbf{k}^2-M^2}{(\mathbf{q}+\mathbf{k})^2+M^2}\right]\right\}.$$

One can verify that the magnetic contributions from diagrams (a) and (b) compensate, as expected. To see this, note that we can write

$$\Pi_{ii}^{(4a)} = (D-1)\int(\mathrm{d}\mathbf{k})\,f(\mathbf{k}) \tag{4.40}$$

and

$$\Pi_{ii}^{(4b)}(0) = \int(\mathrm{d}\mathbf{k})\,k^i\frac{\partial}{\partial k^i}f(\mathbf{k}). \tag{4.41}$$

The sum of the two expressions above vanishes after an integration by parts (allowed by dimensional regularisation). One can also verify that $\Pi_{ii}^{(4a)}(0,\mathbf{p})+\Pi_{ii}^{(4b)}(0,\mathbf{p})$ vanishes as p^2 for $p\to 0$. To do this, it is sufficient to study the contribution of the static modes for $p_0 = 0$ but non-vanishing $\mathbf{p}$ (that is, to replace B_{ii}^{3}, eq. (4.39), with the corresponding contribution for $p \neq 0$). Then, it may be verified that, because of the resummation of thermal masses, the relevant expression admits a well-defined expansion in powers of p^2.

4.4 Order e^4: diagrams 1.c and 1.d

With bare vertices, and zero external momenta, the contributions coming from the diagrams 1.c and 1.d are, respectively,

$$\Pi^c_{\mu\nu}(0) = -4\,(e\mu^\epsilon)^4 \int [\mathrm{d}K] \int [\mathrm{d}Q]\, D_{\mu\nu}(Q) S(K) S(K+Q)\,, \tag{4.42}$$

and

$$\Pi^d_{\mu\nu}(0) = -8\,(e\mu^\epsilon)^4 \int [\mathrm{d}K] \int [\mathrm{d}Q]\, D_{\mu\sigma}(Q)\,(2K^\sigma + Q^\sigma)\, K_\nu\, S^2(K) S(K+Q)\,, \tag{4.43}$$

with $k_0 = i\omega_n$ and $q_0 = i\omega_m$. (The vertex corrections in these diagrams play no role up to the order e^4.) If $\omega_n \neq 0$ and $\omega_m \neq 0$, bare propagators can be used in these expressions in order to obtain the e^4 contribution. If $\omega_n = \omega_m = 0$, the propagators ${}^*S$ and ${}^*D$ should be used instead. Finally, the mixed graph where $\omega_n = 0$ and $\omega_m \neq 0$, or vice-versa, do not contribute to order e^4. In all cases of interest, the calculation is straightforward and leads to the following results:

$$\Pi^{(4c)}_{00}(0) = C^1_{00} + C^2_{00}, \tag{4.44}$$

$$C^1_{00} = 4\,(e\mu^\epsilon)^4 \int [\mathrm{d}K]\,(1-\delta_{n0}) \int [\mathrm{d}Q]'\, S_0(K)\, S_0(K+Q)\, S_0(Q), \tag{4.45}$$

$$C^2_{00} = 4\,(e\mu^\epsilon)^4\; T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{q}^2 + m^2}\, \frac{1}{\mathbf{k}^2 + M^2}\, \frac{1}{(\mathbf{k}+\mathbf{q})^2 + M^2}. \tag{4.46}$$

$$\Pi^{(4c)}_{ii}(0) = C^1_{ii} + C^2_{ii}, \tag{4.47}$$

$$C^1_{ii} = -4(D-1)\,(e\mu^\epsilon)^4 \int [\mathrm{d}K]' \int [\mathrm{d}Q]'\, S_0(K)\, S_0(K+Q)\, S_0(Q), \tag{4.48}$$

$$C^2_{ii} = -4(D-1)\,(e\mu^\epsilon)^4\; T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{q}^2}\, \frac{1}{\mathbf{k}^2 + M^2}\, \frac{1}{(\mathbf{k}+\mathbf{q})^2 + M^2}. \tag{4.49}$$

$$\Pi^{(4d)}_{00}(0) = D_{00} = 12\,(e\mu^\epsilon)^4 \int [\mathrm{d}K] \int [\mathrm{d}Q]\, k_0^2\, S_0^2(K)\, S_0(K+Q)\, S_0(Q). \tag{4.50}$$

$$\Pi^{(4d)}_{ii}(0) = D^1_{ii} + D^2_{ii}, \tag{4.51}$$

$$D^1_{ii} = 8\,(e\mu^\epsilon)^4 \int [\mathrm{d}K]' \int [\mathrm{d}Q]'\, [\mathbf{k}\cdot(\mathbf{2k}+\mathbf{q})]\, S_0^2(K) S_0(K+Q) S_0(Q), \tag{4.52}$$

$$D^2_{ii} = 8\,(e\mu^\epsilon)^4\, T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{q}^2}\, \frac{\mathbf{k}\cdot(\mathbf{2k}+\mathbf{q})}{(\mathbf{k}+\mathbf{q})^2+M^2}\, \frac{1}{\mathbf{k}^2+M^2}. \tag{4.53}$$

It can be easily verified that the sum of the magnetic contributions of diagrams (c) and (d) is a total derivative with respect to $\mathbf{k}$ and therefore vanishes upon integration. For instance

$$\begin{aligned} C^2_{ii} + D^2_{ii} = -4\,(e\mu^\epsilon)^4\, T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{q}^2} \left[(D-1) + k^i \frac{\partial}{\partial k^i} \right] & \\ \left(\frac{1}{\mathbf{k}^2+M^2}\, \frac{1}{(\mathbf{k}+\mathbf{q})^2+M^2} \right) &= 0, \end{aligned} \tag{4.54}$$

and similarly $C^1_{ii} + D^1_{ii} = 0$. One can also verify that the function $\Pi^{(4c)}_{\mu\nu}(0,p) + \Pi^{(4d)}_{\mu\nu}(0,p)$ is analytic in p^2 as $p \to 0$.

4.5 Electric mass to order e^4

By adding together the results of sections 3.3 and 3.4, we are now able to obtain the correction of order e^4T^2 to $\Pi_{00}(0,0)$. The relevant equations are (4.30)–(4.32), (4.34)–(4.35), (4.45)–(4.46) and (4.50), which express the relevant contributions of the four diagrams in Fig. 1. A simple inspection of these equations shows that the following compensations arise:

$$A_2 + C^1_{00} = 0, \qquad\qquad B^2_{00} + D_{00} = 0. \tag{4.55}$$

That is, the only non-trivial contribution of the *non-static* two-loops diagrams is that given by $A_1 + B^1_{00}$, eqs. (4.30) and (4.34). It is proportional to the scalar thermal mass squared M^2:

$$\begin{aligned} A_1 + B^1_{00} &= 2\,(e\mu^\epsilon)^2\, M^2 \int [\mathrm{d}K]'\, S_0^2(K)\, \Big(1 + 4k_0^2\, S_0(K)\Big) \\ &= 2M^2(D-4)\,(e\mu^\epsilon)^2\; b_2\,, \end{aligned} \tag{4.56}$$

where the second line follows from the first one after an integration by parts (see eq. (3.7)). By evaluating b_2 from eq. (3.7), we obtain

$$A_1 + B^1_{00} \;=\; -\left(\frac{eM}{2\pi}\right)^2 \;=\; -\left(\frac{e^2T}{4\pi}\right)^2 \tag{4.57}$$

for the contribution of the non-static modes. The contribution to $\Pi^{(4)}_{00}(0,0)$ which arises after resumming thermal masses in purely static two-loops diagrams involves the terms A_3 and C^2_{00} (eqs. (4.32) and (4.46)), whose sum is computed in Appendix B:

$$\begin{aligned} A_3 + C^2_{00} &= -\left(\frac{e^2T}{4\pi}\right)^2 \left(\frac{m-M}{M} + 4\ln\frac{m+2M}{2M}\right) \\ &= -\left(\frac{e^2T}{4\pi}\right)^2 \left(\frac{2}{\sqrt{3}} - 1 + 4\ln\left(1+\frac{1}{\sqrt{3}}\right)\right). \end{aligned} \tag{4.58}$$

By adding eqs. (4.57) and (4.58), we derive finally

$$\begin{aligned} \Pi^{(4)}_{00}(0,0) &= -\left(\frac{e^2T}{4\pi}\right)^2 \left(\frac{m}{M} + 4\ln\frac{m+2M}{2M}\right) \\ &= -\left(\frac{e^2T}{4\pi}\right)^2 \left(\frac{2}{\sqrt{3}} + 4\ln\left(1+\frac{1}{\sqrt{3}}\right)\right). \end{aligned} \tag{4.59}$$

No UV divergencies have been encountered in the derivation of this result: all the terms which survive the compensations (4.55) are UV finite. Note that the first line vanishes if $m=0$, i.e. if we "forget" to resum the static longitudinal photon lines.

The result (4.59) represents the main ingredient for computing the correction of order e^4 to the Debye screening mass m_D^2. In order to solve the pole equation (1.2) to this accuracy, we actually need the electrostatic polarization operator $\Pi_{00}(0,p)$ for $p^2=-m^2$ up to order e^4. That is, we still have to evaluate the momentum dependence of $\Pi_{00}(0,p)$ with the prescribed accuracy. Because of the restriction to soft momenta ($p=im\sim eT$), this can be easily done: At one-loop order, we have $\Pi^{1L}_{00}(0,p) = -m^2 - a\, e^2p^2 + \mathcal{O}(e^2p^4/T^2)$, with the coefficient a computed in App. A (see eqs. (B.7)–(B.8)). At two-loop order, only the momentum dependence of the *static* diagrams is relevant for $\Pi_{00}(0,p)$ to order e^4 (for non-static graphs, the leading non-trivial behaviour at small momenta $p\ll T$ is $\sim e^4p^2$, which is of order e^6T^2 for $p\sim eT$). The only static two loop graph which is momentum dependent is the diagram 1.c, which gives the following contribution to $\Pi_{00}(0,p)$:

$$C^2_{00}(p) = 4\,(e\mu^\epsilon)^4\, T^2 \int(\mathrm{d}\mathbf{k})\int(\mathrm{d}\mathbf{q})\, \frac{1}{(\mathbf{q}+\mathbf{p})^2+m^2}\,\frac{1}{\mathbf{k}^2+M^2}\,\frac{1}{(\mathbf{k}+\mathbf{q})^2+M^2}. \tag{4.60}$$

(For $p=0$, this obviously reduces to C^2_{00}, eq. (4.46).) It is easy to see that the momentum dependence of $C^2_{00}(p)$ *does* matter to order e^4 if $p\sim eT$: even if eq. (4.60) is indeed analytic with respect to p^2 as $p\to 0$, the thermal masses which ensure this analitycity are precisely of order eT; it follows that, for momenta $p\sim eT$, all the terms in the small momentum expansion are of the same order and the exact expression for $C^2_{00}(p)$ should be used.

When the previous results are set together, we obtain, for $p \sim eT$,

$$\Pi_{00}(0,p) = -m^2 - a\,e^2p^2 - \delta Z_3 p^2 + \Pi_{00}^{(3)}(0,0) + \Pi_{00}^{(4)}(0,0) \\ + \Big(C_{00}^2(p) - C_{00}^2(0)\Big) + \mathcal{O}(e^5T^2). \tag{4.61}$$

The counterterm

$$\delta Z_3 = -\frac{e^2}{3(4\pi)^2}\,\frac{1}{\epsilon} \tag{4.62}$$

accounts for the photon wave-function renormalization in the minimal substraction scheme and cancels the divergent piece of a in the limit $D \to 4$ (see eq. (B.8)). This leaves the following renormalised value for the coefficient a:

$$a_R = \frac{1}{3(4\pi)^2}\left(\gamma + 2 + \ln\frac{\mu^2}{4\pi T^2}\right) \tag{4.63}$$

To order e^4, the solution to eq. (1.2) is

$$m_D^2 = m^2(1 - a_R e^2) - \Pi_{00}^{(3)}(0,0) - \Pi_{00}^{(4)}(0,0) - \Big(C_{00}^2(im) - C_{00}^2(0)\Big) + \mathcal{O}(e^5T^2). \tag{4.64}$$

The difference $C_{00}^2(p = im) - C_{00}^2(0)$ is readily computed from eqs.(4.60) and (4.46) as

$$C_{00}^2(p = im) - C_{00}^2(0) = \left(\frac{e^2T}{2\pi}\right)^2 \int_0^\infty \frac{\mathrm{d}x}{x}\,\mathrm{e}^{-(2M+m)x}\left(\frac{\sinh mx}{mx} - 1\right) \\ = \left(\frac{e^2T}{2\pi}\right)^2\left(1 - \frac{M}{m}\ln\frac{2M+m}{2M} - \frac{M+m}{m}\ln\frac{2(M+m)}{2M+m}\right). \tag{4.65}$$

By also using eqs. (4.9) and (4.59) for $\Pi_{00}^{(3)}(0,0)$ and $\Pi_{00}^{(4)}(0,0)$, we finally obtain

$$m_D^2 \;=\; T^2\left(\frac{e^2}{3} - \frac{e^3}{4\pi} + b\,\frac{e^4}{(2\pi)^2} - \frac{e^4}{(12\pi)^2}\left[\gamma + 2 + \ln\frac{\mu^2}{4\pi T^2}\right]\right) \;+\; \mathcal{O}(e^5T^2), \tag{4.66}$$

where b is the positive number

$$b \;=\; -1 + \frac{1}{2\sqrt{3}} + \left(1 + \frac{\sqrt{3}}{2}\right)\ln\left(1 + \frac{2}{\sqrt{3}}\right) \;=\; 0.7211328\ldots, \tag{4.67}$$

and e is the gauge coupling at the scale μ ($e \equiv e(\mu)$), as defined by the minimal substraction scheme. The expression (4.66) is independent of the choice of μ if $\mathrm{d}e/\mathrm{d}\ln\mu = e^3/48\pi^2$, which is precisely the one-loop $\beta-$function of scalar QED. To verify that the screening mass (4.66) is actually gauge independent, we rely on the gauge invariance of $\Pi_{00}(0,0)$ and on the fact that the momentum dependent terms which enter $\Pi_{00}(0,p)$ to order e^4 (see eq. (4.61)) arise from gauge-independent diagrams. (The latter assertion is obvious for the one-loop contribution, $\Pi_{00}^{1L}(0,p)$; as for the two-loop contribution of eq. (4.65), this only involves the time-time piece of the static photon propagator, which is the same in all covariant gauges.)

4.6 Summary: effective theory for static modes

In the previous sections, we have computed the zero-momentum limit ($\omega = 0$, $p \to 0$) of the photon polarization tensor to order e^4. There are two type of terms occuring in the perturbative expansion, and we want to discuss this in more detail now. Consider the electric piece, $\Pi_{00}(0,0)$, as given to order e^4 by eqs. (2.5), (4.9) and (4.59). Some of the contributions in these equations arise from *purely non-static* one- and two-loop graphs, which are computed with the standard Feynman rules (i.e. without resummation), except for the fact that the terms with $\omega_n = 0$ are excluded from the sums over Matsubara frequencies. This is the case for the leading order electric mass, (2.5), and also for the two-loop contribution of eq. (4.57). Since no IR problem arises in such diagrams, naive power counting applies. Quite generally then, n-loop diagrams with no static internal line will contribute to $\Pi_{00}(0,0)$ to order e^{2n}. The second type of terms involve one- and two-loop diagrams in which *all* the internal lines are static and include the corresponding thermal masses. The non-analytic contribution of order e^3, eq. (4.9), belongs to this category, but this is also the case for the order-e^4 contribution of eq. (4.58). As already mentioned, the *mixed* two-loop graphs (where one of the internal frequencies is vanishing, and the other is not) contribute to order e^5. In general, it can be verified by power counting that a n-loop diagram ($n \geq 1$) with only static (dressed) internal lines contributes to $\Pi_{00}(0,0)$ to the order e^{n+2}, and eventually dominates over the corresponding non-static contribution (of order e^{2n}) as soon as $n \geq 3$. As for the mixed n-loop graphs (with $n \geq 2$), their leading contribution to $\Pi_{00}(0,0)$ is also of the order e^{n+2} for $n \geq 3$ (it is only of the order $e^5 \equiv e^{n+3}$ for $n = 2$).

A different, probably more transparent, way to look at the perturbative expansion after the resummation of the thermal masses is to consider, as an intermediate step, the effective three-dimensional theory for static and long wavelength ($p \lesssim eT$) fields. This is obtained after integrating non-static loops with static external lines[20]. The lagrangian of the effective theory reads ($D_i \equiv \partial_i - ie_3 A_i$)

$$\begin{aligned}\mathcal{L}_{eff} = \frac{1}{4}F_{ij}^2 + \frac{1}{2\alpha}(\partial_i A_i)^2 + \frac{1}{2}\,(\partial_i A_0)^2 + \frac{1}{2}\,m_0^2\,A_0^2 + e_3^2 A_0^2 \phi^\dagger \phi \\ +(D_i\phi)^\dagger(D_i\phi) + M_0^2\phi^\dagger\phi + \delta\mathcal{L}.\end{aligned} \tag{4.68}$$

The magnetostatic gauge fields $A_i(\mathbf{x})$, $i = 1,\ 2,\ 3$, the electrostatic (gauge-invariant) field $A_0(\mathbf{x})$, and the scalar field $\phi(\mathbf{x})$, may be identified, up to normalizations, with the zero-frequency components of the original fields. The term $\delta\mathcal{L}$ contains higher order vertices, but also derivative corrections to the n-point vertices shown explicitly; in particular, it contains the counterterms necessary for UV renormalization. All these operators, as well

as the parameters e_3, m_0^2 and M_0^2, are obtained as power series in e^2 by evaluating non-static diagrams with static external lines in the original theory, and by expanding with respect to the external momenta. To leading order, $e_3^2 \approx e^2 T$, $m_0^2 \approx m^2$ and $M_0^2 \approx M^2$. Generally, a vertex with n static external lines which is absent at the tree-level is first induced at the order e^n, via non-static one-loop graphs. Actually, some particular vertices may be induced at a level higher than e^n, or may even vanish, because of some symmetry. The lagrangian (4.68) is invariant under static gauge transformations.

The loop corrections in the effective theory generate all the static diagrams of the original theory. They also include the *mixed* graphs: indeed, a non-static subgraph of an original mixed diagram appears as a bare vertex in the effective theory, while a static subgraph appears as a loop correction. The calculation that we have done before can be understood simply in terms of this reorganized perturbation theory. Consider for exemple the calculation of $\Pi_{00}(0,0)$. In the effective theory, we have $\Pi_{00}(0,0) \equiv -m_0^2 - \Sigma_{A_0}(\mathbf{p}=0)$, v where $\Sigma_{A_0}(\mathbf{p})$ is the self-energy of the field A_0 in the effective theory. Thus, to get $\Pi_{00}(0,0)$ to order e^4, we need both m_0^2 and $\Sigma_{A_0}(0)$ to order e^4. We already know the result for m_0^2: by evaluating the corresponding non-static one- and two-loop graphs, we have obtained (see eqs. (2.5) and (4.57))

$$m_0^2 = \frac{e^2 T^2}{3} \left\{ 1 + \frac{3}{(4\pi)^2} e^2 + \mathcal{O}(e^4) \right\} . \tag{4.69}$$

To order e^4, $\Sigma_{A_0}(0)$ is given by the one- and two-loop diagrams in Fig. 8, where the leading order vertices ($e_3^2 = e^2 T$) and massive propagators should be used ($M_0^2 = M^2$ and $m_0^2 = m^2$). Their evaluation is straightforward, and leads to

$$\Sigma_{A_0}(0) = -\frac{e^3 T^2}{4\pi} + \left(\frac{e^2 T}{4\pi}\right)^2 \left(\frac{2}{\sqrt{3}} - 1 + 4 \ln\left(1 + \frac{1}{\sqrt{3}}\right)\right) , \tag{4.70}$$

which reproduces the contribution of the relevant static graphs in the original theory. By adding the contributions (4.69) and (4.70), we recover our previous result for $\Pi_{00}(0,0)$.

It is easy to see the systematics of the higher order corrections. The perturbative expansion of $\Pi_{00}(0,0)$ decomposes into the sum of two terms: the first one, m_0^2, is a power series in e^2 and arises from the non-static diagrams of the original theory; the second term, $\Sigma_{A_0}(\mathbf{p}=0)$, is obtained as a power series in e in the effective theory. The fact that it is e rather than e^2 which governs the perturbative expansion in the effective theory is due to the fact that the loop integrals in the effective theory would be IR divergent in the absence of the thermal masses. Thus, a n-loop diagram contributing to Σ_{A_0} has an explicit factor of $e_3^{2n} \sim e^{2n} T^n$; since it has dimension two, it is proportional to $e^{2n} T^2 (T/M)^{n-2} \sim (e^4 T^2) e^{n-2}$, where $M \sim eT$ is any of the bare masses m_0 or M_0,

or the external momentum, playing the role of an infrared cut-off. Since $n \geq 1$, we see emerging a power expansion in e, starting at the order e^3. We have assumed here that all the vertices entering the n-loop diagram are of the type explicitly depicted in eq. (4.68), i.e. vertices which are already present in the original theory at the tree-level. Actually, the new vertices contained in $\delta\mathcal{L}$ will also contribute, but only starting at the order e^5 (as may be verified by power counting). In particular, to obtain the electric screening mass to order e^5 we need the bare parameters m_0^2 and M_0^2 to two-loop order in the original theory (e.g. eq. (4.69) for m_0^2) and the corrections to $\Sigma_{A_0}(\mathbf{p}=0)$ up to three-loop order in the effective theory. The latter will involve one-loop diagrams with some of the 4-point effective vertices from $\delta\mathcal{L}$.

5 Conclusion

In this paper we computed directly the Debye screening masses for QED and SQED two orders above their lowest nontrivial values and verified explicitly that the corresponding magnetic masses vanished in the same approximation. It was also shown that the results were both gauge and renormalization group invariant. For QED, we further argued that the vanishing of the magnetic mass holds to all orders in perturbation theory: Since the (odd) Matsubara frequencies of the fermionic propagators play the role of a mass, the static photon lines are always coupled to massive fields; there is no closed loop of massless fields. This is enough to ensure the analyticity with respect to p^2 (p being the external three momentum) of the Feynman diagrams contributing to the static polarisation tensor. These arguments can be extended to SQED as well, in spite of the fact that the propagator of the scalar particle may be at zero Matsubara frequency: Once thermal masses of order gT are given to the charged particles and to the longitudinal photons by the appropriate resummations, the situation in SQED becomes similar to that of QED —- the magnetostatic fields, which remain massless, couple only to massive particles. By contrast, the properties of the polarisation tensor are different in non-abelian theories. For example, in QCD there is no identity like (1.4) and, in fact, the self-interactions of the magnetostatic gluons are believed to generate a magnetic screening mass $\sim g^2T$ [24].

In QED, the above arguments on the analyticity, with respect to p^2, of the static polarisation tensor are valid for momenta p smaller than T because the odd fermionic frequencies $\sim T$ provide the relevant infrared cut-off. This is why, for example, in eq. (3.21) we could use the small momentum expansion of $\Pi_{00}(0,p)$ up to $p \sim im$. On the other hand, in SQED, the infrared cut-off is provided by the screening masses $\sim gT$, and

the small p^2 expansion holds only for $p < gT$. Thus for example, in the calculation of the electric mass at order e^4, we could not expand $\Pi_{00}(0,p)$ for $p \sim im$ (see eq.(4.65)). In fact singularities do occur for imaginary values of p of order gT. For exemple, ${}^*\Pi_{ii}(0,p)$ in eq. (4.12) has branch point singularities at $p = \pm 2iM$, and ${}^*\Sigma(0,p)$, eq. (4.14), has similar singularities for $p = \pm iM$. It is interesting to note, however, that while ${}^*\Pi_{ii}(0,p = \pm 2iM) = 0$, ${}^*\Sigma(0,p = \pm iM)$ diverges. This latter behaviour prevents the calculation of the correction of order e^3T^2 to the screening mass M^2 of the scalar field along the lines followed in this paper for m_D^2.

An analytical structure similar to that of the scalar self-energy ${}^*\Sigma(0,p)$ has been observed in the calculation of the non-abelian Debye mass at the next to leading order, i.e. m_D^2 up to order g^3T^2 [3, 23]. This analogy between the scalar self-energy in SQED and the gluon self-energy in QCD can be expected from the similarity between the charged scalar sector of $\mathcal{L}_{eff}$, eq. (4.68), and the electrostatic sector of the corresponding three-dimensional effective theory for high temperature QCD[20, 21]. In Refs. [3, 23], the logarithmic divergence in the electrostatic gluon polarisation tensor at $p = im$ has been cut-off by giving the static gluon a magnetic mass of order g^2T. Because of the analogous difficulty in SQED where no magnetic mass can be generated, we feel that it is worthwhile to look for alternative treatement of this problem[26].

We had stated that the derivation of (1.3) was formal but checked its correctness explicitly to fifth order in QED. On the other hand if one has faith in that identity, then our verification may be reinterpreted as actually having provided a nontrivial check on the correctness of our resummed perturbation expansion. For SQED we did not verify the identity to the fourth order but in this case we repeated our computations using two different resummation schemes, thus providing again some useful cross checks.

Of course, static correlators are not the only relevant probes of a plasma. The plasma frequency and damping rate are two important quantities which can be deduced only from dynamic correlation functions. Compared to static quantities, the computation of dynamic quantities is more involved — not only must one work in real-time (that is an analytical continuation of imaginary time, or the real-time formalism itself), but one also has to use the full machinery developed in [4] in the resummation necessary for higher order calculations. As far as we know, of the four possible explicit resummation methods mentioned in the Introduction, only method (c) (use of rearranged lagrangians incoporating the hard thermal loop effects) has so far been used for dynamical calculations beyond leading order [4, 17, 25, 10].

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Appendix A

In this Appendix, we review a derivation of the hard thermal loop for the photon polarization tensor which emphasizes its classical character and the ensuing universality of its structure for any gauge theory. To start with, we recall that $\Pi_{\mu\nu} = \delta j_\mu^{ind}/\delta A^\nu$, where j_μ^{ind} is the current induced by the electromagnetic field. To leading order in e

$$j_\mu^{ind}(x) = e\int \frac{\mathrm{d}^4K}{(2\pi)^4}\, 2k^\mu\, W(K,x) = e\int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \frac{k^\mu}{\epsilon_k}\, [\delta N_+(\mathbf{k},x) - \delta N_-(\mathbf{k},x)]\,, \tag{A.1}$$

where $W(K,x) = 2\pi\,\delta(k_0^2 - \epsilon_k^2)\;[\theta(k_0)\delta N_+(\mathbf{k},x) + \theta(-k_0)\delta N_-(-\mathbf{k},x)]$ is a gauge-invariant Wigner function for the charged particles, ultimately related to the scalar propagator in the presence of the gauge fields[11]. The δ-function defines the mass-shell for the scalar particles; in the present approximation, it coincides with the *free* mass-shell, $k_0^2 = \epsilon_k^2$ (with $\epsilon_k = k$ for massless particles). The equation satisfied by the Wigner functions $\delta N_\pm(\mathbf{k},x)$ is[11]

$$(v\cdot\partial_x)\;\delta N_\pm(\mathbf{k},x) = \mp\, e\,\mathbf{v}\cdot\mathbf{E}(x)\,\frac{dN_0}{d\epsilon_k}, \tag{A.2}$$

where $v^\mu \equiv (1,\mathbf{v})$, $\mathbf{v} \equiv \mathbf{k}/\epsilon_k$, $v\cdot\partial_x = \partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}$, $\mathbf{E}$ is the mean electric field, and $N_0(\epsilon_k) = 1/(\exp(\beta\epsilon_k) - 1)$ is the equilibrium occupation factor for bosons. Note that for an abelian plasma this is merely the (linearized) Vlasov equation. By combining eqs. (A.1) and (A.2), one finds $j_\mu^{ind}(\omega,\mathbf{p}) = \Pi_{\mu\nu}(\omega,\mathbf{p})\,A^\nu(\omega,\mathbf{p})$, with

$$\Pi_{\mu\nu}(\omega,\mathbf{p}) \;=\; 2e^2\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{dN_0}{d\epsilon_k}\left(\delta_{\mu 0}\delta_{\nu 0} - v_\mu v_\nu\,\frac{\omega}{\omega - \mathbf{v}\cdot\mathbf{p}}\right). \tag{A.3}$$

For $\epsilon_k = k$, the radial integral over k can be easily performed, and one recovers the expression (2.4).

In abelian plasmas, j_μ^{ind} is gauge-invariant and linear in A_μ, as shown by eqs. (A.1) and (A.2), so that there is no hard thermal loop for the n-photon vertices with $n > 2$.

In contrast, the QCD induced current is a *color* vector, i.e. j^a_μ transforms in the adjoint representation of $SU(N)$. Then, as required by gauge symmetry, the kinetic equation (A.2) involves a covariant line-derivative (i.e. $v \cdot \partial_x$ is replaced by $v \cdot D_x$ in its l.h.s.), so that the color current is a *non-linear* functional of the gauge potentials. It follows then that the higher functional derivatives of j^a_μ with respect to A^a_μ are also non-vanishing, and define hard thermal loops for vertex functions with an arbitrary number of soft external gluon lines. However, for weak fields, $|A_\mu| \to 0$, the kinetic equation (A.2) is formally the same for abelian and non-abelian plasmas. This explains why the expressions obtained for the polarization tensor in this approximation are, up to a trivial factor which counts the relevant degrees of freedom, identical for QED, QCD or scalar QED.

As for the hard thermal loop for the scalar self-energy, this describes the response of the plasma to long-wavelength scalar mean fields. Since according to eq. (4.2), $\Sigma^{(2)}$ is a *local* operator, gauge covariance requires this operator to be independent of the gauge mean field[11], so that there are no hard thermal loops for vertices involving scalar and photon external lines. We conclude that the self-energy corrections in eqs. (2.4) and (4.2) are the *only* hard thermal loops for SQED, in accordance with the power counting arguments of Refs. [4, 10].

The previous derivation of $\Pi_{\mu\nu}$ can be generalized to take into account the thermal mass M aquired by the charged particles. Since $M \ll T$, the only effect is the modification of the mass shell condition which becomes $k_0^2 = \epsilon_k^2 \equiv k^2 + M^2$. After inserting this into eq. (A.3), one obtains

$$\Pi_{\mu\nu}(\omega = 0, \mathbf{p}) \;=\; -m^2(T, M)\, \delta_{\mu 0}\delta_{\nu 0}, \tag{A.4}$$

with

$$m^2(T, M) = -\,2e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{dN_0}{d\epsilon_k} \;=\; \frac{e^2T^2}{3} \;-\; \frac{e^2MT}{4\pi} \;+ \mathcal{O}\left(e^2M^2\right). \tag{A.5}$$

As advertised in section 3.2, this is the correct electric mass up to the order e^3.

Appendix B

In this appendix, we derive some formulae which are referred to in Sect. 3. We consider first the small momentum behaviour of the one-loop static polarization tensor,

$\Pi^{1L}_{\mu\nu}(0,\mathbf{p})$. For the magnetic sector, the diagrams in Fig. 3 imply

$$\Pi^{1L}_{ii}(0,\mathbf{p}) = (e\mu^\epsilon)^2 \int [\mathrm{d}K] \left\{ \frac{2(D-1)}{\omega_n^2+\mathbf{k}^2} - \frac{(\mathbf{2k}+\mathbf{p})^2}{(\omega_n^2+\mathbf{k}^2)\,(\omega_n^2+(\mathbf{k}+\mathbf{p})^2)} \right\}. \tag{B.1}$$

The dominant contribution as $p\to 0$ is given by the $\omega_n = 0$ term in the Matsubara sum and it is linear in p:

$$\begin{aligned} \left[\Pi^{1L}_{ii}(0,\mathbf{p})\right]_{n=0} &= (e\mu^\epsilon)^2\, \mathbf{p}^2\, T \int (\mathrm{d}\mathbf{k})\, \frac{1}{\mathbf{k}^2}\, \frac{1}{(\mathbf{k}+\mathbf{p})^2} \\ &= \frac{e^2 pT}{4\pi^2} \int_0^\infty \frac{dx}{x}\, \ln\left|\frac{x+1}{x-1}\right| \;=\; \frac{1}{8}\, e^2 pT. \end{aligned} \tag{B.2}$$

For the time-time component we obtain

$$\begin{aligned} \Pi^{1L}_{00}(0,\mathbf{p}) &= -\,(e\mu^\epsilon)^2 \int [\mathrm{d}K] \left\{ \frac{2}{\omega_n^2+\mathbf{k}^2} - \frac{4\omega_n^2}{(\omega_n^2+\mathbf{k}^2)\,(\omega_n^2+(\mathbf{k}+\mathbf{p})^2)} \right\} \\ &= -m^2 + (e\mu^\epsilon)^2 \int [\mathrm{d}K] \frac{4\omega_n^2}{(\omega_n^2+\mathbf{k}^2)} \left(\frac{1}{\omega_n^2+(\mathbf{k}+\mathbf{p})^2} - \frac{1}{\omega_n^2+\mathbf{k}^2} \right), \end{aligned} \tag{B.3}$$

where

$$m^2 \equiv -\Pi^{1L}_{00}(0,0) = 2(D-2)\,(e\mu^\epsilon)^2 \int [\mathrm{d}Q]\, S_0(Q) = \frac{e^2T^2}{3}. \tag{B.4}$$

Note that the static mode $\omega_n = 0$ does not contribute to the last sum in eq. (B.3); accordingly, one can expand the denominator for small p without generating IR singularities:

$$\frac{1}{\omega_n^2+(\mathbf{k}+\mathbf{p})^2} = \frac{1}{\omega_n^2+\mathbf{k}^2} \left\{ 1 - \frac{\mathbf{p}^2+2\mathbf{k}\cdot\mathbf{p}}{\omega_n^2+\mathbf{k}^2} + \frac{(\mathbf{p}^2+2\mathbf{k}\cdot\mathbf{p})^2}{(\omega_n^2+\mathbf{k}^2)^2} - \ldots \right\}. \tag{B.5}$$

By keeping only the terms quadratic in p, we obtain

$$\Pi^{1L}_{00}(0,\mathbf{p}) \;=\; -m^2 - a\, e^2\, p^2 + \mathcal{O}(e^2p^4/T^2), \tag{B.6}$$

where

$$\begin{aligned} a &\equiv 4\mu^{2\epsilon} \int [\mathrm{d}K] \frac{\omega_n^2}{(\omega_n^2+\mathbf{k}^2)^3} \left(1 - \frac{4}{D-1}\, \frac{\mathbf{k}^2}{\omega_n^2+\mathbf{k}^2} \right) \\ &= \frac{5-D}{3} \mu^{2\epsilon} \int [\mathrm{d}K]' S_0^2(K). \end{aligned} \tag{B.7}$$

and some integrations by parts have been necessary to get the last line. By also using eq. (3.7) with $n = 2$ and by expanding in ϵ, we obtain

$$a = \frac{1}{3(4\pi)^2} \left(\frac{1}{\epsilon} + \gamma + 2 + \ln\frac{\mu^2}{4\pi T^2} + \mathcal{O}(\epsilon) \right). \tag{B.8}$$

Finally, we turn to the evaluation of the integrals (4.32) and (4.46) which contribute to $\Pi_{00}(0)$ to order e^4. We organize this computation as follows:

$$A_3 + C_{00}^2 = a_1 + a_2 + a_2, \tag{B.9}$$

where

$$a_1 \equiv 2\,(e\mu^\epsilon)^4\, T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{(\mathbf{k}^2+M^2)^2} \left(\frac{1}{\mathbf{q}^2+m^2} + \frac{1}{\mathbf{q}^2+M^2} - \frac{2}{\mathbf{q}^2} \right), \tag{B.10}$$

$$a_2 \equiv 8\,(e\mu^\epsilon)^4\, M^2 T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{q}^2}\, \frac{1}{(\mathbf{k}^2+M^2)^2}\, \frac{1}{(\mathbf{q}+\mathbf{k})^2+M^2}, \tag{B.11}$$

and

$$a_3 \equiv -4\,(e\mu^\epsilon)^4\, T^2 \int (\mathrm{d}\mathbf{k}) \int (\mathrm{d}\mathbf{q})\, \frac{1}{\mathbf{k}^2+M^2}\, \frac{1}{(\mathbf{q}+\mathbf{k})^2+M^2} \left(\frac{1}{\mathbf{q}^2} - \frac{1}{\mathbf{q}^2+m^2} \right). \tag{B.12}$$

All the integrals above are UV and IR finite in $D=4$, and thus we can set $\epsilon = 0$. The evaluation of a_1 is straightforward, with the result

$$a_1 = -\left(\frac{e^2 T}{4\pi}\right)^2 \frac{m+M}{M}. \tag{B.13}$$

In order to compute a_2 and a_3, it is convenient to use the coordinate-space representation of the static propagators, by writing

$$\frac{1}{k^2+M^2} = \int \mathrm{d}^3x\, \mathrm{e}^{i\vec{k}\cdot\vec{x}}\, \frac{\mathrm{e}^{-Mx}}{4\pi x}, \qquad \frac{1}{(k^2+M^2)^2} = \frac{1}{8\pi M} \int \mathrm{d}^3x\, \mathrm{e}^{i\vec{k}\cdot\vec{x}}\, \mathrm{e}^{-Mx}, \tag{B.14}$$

with $x \equiv |\vec{x}|$. Then one obtains

$$a_2 = 2\left(\frac{e^2 T}{4\pi}\right)^2, \tag{B.15}$$

and

$$a_3 = -4\left(\frac{e^2 T}{4\pi}\right)^2 \int_0^\infty \frac{\mathrm{d}x}{x} \left(\mathrm{e}^{-2Mx} - \mathrm{e}^{-(2M+m)x}\right) = -4\left(\frac{e^2 T}{4\pi}\right)^2 \ln\frac{m+2M}{2M}. \tag{B.16}$$

By adding together eqs. (B.13), (B.15) and (B.16), we obtain the expression (4.58).

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Figure captions

Figure 1. The skeleton diagrams for the self-energy of the photon in scalar QED.

Figure 2. The two loop contributions to the photon self energy in QED.

Figure 3. Two loop diagrams in QED in which the photon internal line is dressed with the thermal mass.

Figure 4. One loop contributions to the polarisation tensor in SQED.

Figure 5. One loop contributions to the scalar self-energy in SQED.

Figure 6. One loop contributions to the polarisation tensor in SQED, with the static internal lines dressed by thermal masses.

Figure 7. One loop contributions to the scalar self-energy in SQED, with the static internal lines dressed by thermal masses.

Figure 8. The self-energy of the static photon in the three-dimensional efffective theory (see eq.(4.70)). Diagram (c) gives no contribution in dimensional regularisation. Dashed line: longitudinal photon. Wavy line: transverse photon. Full line: scalar field.